Definition A formula F is $\mathcal{S}$-inconsistent if there is a derivation
$$\begin{array}{c} F \\ \| \mathcal{S} \\ \mathsf{false} \end{array}.$$

A set of formulae Γ is $\mathcal{S}$-inconsistent if for some $\{F_1 \ldots F_n\} \subseteq \Gamma$ there is a derivation
$$\begin{array}{c} (F_1, \ldots, F_n) \\ \| \mathcal{S} \\ \mathsf{false} \end{array}.$$

Definition System KS a.k.a KS$\downarrow$ includes all down-rules and all self-dual rules of system SKS i.e.

$$KS\downarrow = KS = \{ i\downarrow, s, c\downarrow, w\downarrow \}$$

System KS$\uparrow$ includes all up-rules and all self-dual rules, i.e.

$$KS\uparrow = \{ i\uparrow, s, c\uparrow, w\uparrow \}.$$

Definition max. $\mathcal{S}$-consistent : as usual.

Theorem KS$\downarrow$ is complete.

Proof

F is valid $\overset{①}{\Rightarrow}$ $\bar{F}$ is unsatisfiable $\overset{②}{\Rightarrow}$ $\bar{F}$ is KS$\uparrow$-inconsistent $\overset{③}{\Rightarrow}$ F is KS$\downarrow$-provable.

① is immediate, ③ by flipping, ② is the contrapositive of the next th[illegible]

Lemma 1 If Γ is $KS\uparrow$-consistent then there is a $\Gamma^* \supseteq \Gamma$ s.t. Γ^* is max. $KS\uparrow$-consistent.

Proof the usual 'enumeration (technique) of formulae etc., gives a non-decreasing sequence of sets $\Gamma_1 \dots \Gamma_n \dots$ with $\Gamma_1 = \Gamma$

$$\Gamma^* = \bigcup_n \Gamma_n$$

Γ_n is consistent by def.

$\boxed{\text{consistent} = KS\uparrow\text{-consistent}}$

Γ^* is consistent, because if it were not, then for some $\{F_1 \dots F_K\} \subseteq \Gamma^*$ we'd have

$$\begin{array}{c} (F_1, \dots, F_K) \\ \| \, KS\uparrow \\ \text{false} \end{array} \quad \text{from which we could build} \quad \begin{array}{c} \wedge \Gamma_n \\ \| \, w\uparrow \\ (F_1, \dots, F_K) \\ \| \, KS\uparrow \\ \text{false} \end{array},$$

but Γ_n is consistent, which is a contradiction.

Γ^* is max. consistent, because suppose $\Gamma^* \cup \{A\}$ is consistent. Then for all n, $\Gamma_n \cup \{A\}$ is consistent. Thus $A \in \Gamma_n$ for some n. Thus $A \in \Gamma^*$.

$\square$

Theorem If a formula F is $KS{\uparrow}$-consistent, then it is satisfiable.

Proof Let Γ^* be the max. consistent extension of $\{F\}$, which exists by Lemma 1.

Define $\sigma(a) = \begin{cases} \text{true} & \text{if } a \in \Gamma^* \\ \text{false} & \text{otherwise} \end{cases}$ (a is an atom.)

Claim: $A \in \Gamma^* \Longrightarrow \sigma \models A$

Ind. on A:

base cases: 1. A is an atom a. Claim holds by Def. of σ.

2. A is a negative atom $\bar{a}$.

$$\bar{a} \in \Gamma^* \xRightarrow[\text{Lemma 2}]{} a \notin \Gamma^* \Rightarrow \sigma \models \bar{a}.$$

ind. cases: 3. A is a conjunction (A_1, A_2).

$$(A_1, A_2) \in \Gamma^* \xRightarrow[\text{Lemma 3}]{} A_1 \in \Gamma^* \text{ and } A_2 \in \Gamma^*$$

$$\xRightarrow[\text{IH}]{} \sigma \models A_1 \text{ and } \sigma \models A_2 \Rightarrow \sigma \models (A_1, A_2).$$

4. A is a disjunction $[A_1, A_2]$.

$$[A_1, A_2] \in \Gamma^* \xRightarrow[\text{Lemma 4}]{} A_1 \in \Gamma^* \text{ or } A_2 \in \Gamma^*$$

$$\xRightarrow[\text{IH}]{} \sigma \models A_1 \text{ or } \sigma \models A_2 \Rightarrow \sigma \models [A_1, A_2].$$

Lemma 2 $\quad \bar{a} \in \Gamma^* \Rightarrow a \notin \Gamma^*.$

Proof: Assume both $\bar{a} \in \Gamma^*$ and $a \in \Gamma^*$.

Then $\{a, \bar{a}\} \subseteq \Gamma^*$ and $\boxed{\mathsf{ai}\uparrow \dfrac{(a,\bar{a})}{\mathsf{false}}}$,

but Γ^* is $\mathsf{KS}\uparrow$-consistent. Contradiction. $\square$

Lemma 3 $\quad (A_1, A_2) \in \Gamma^* \Rightarrow A_1 \in \Gamma^*$ and $A_2 \in \Gamma^*$

Proof: Assume $(A_1, A_2) \in \Gamma^*$ and $A_1 \notin \Gamma^*$. Then there is a set $\{B_1, \ldots, B_n\} \subseteq \Gamma^*$ s.t.

$$\begin{array}{c} (B_1, \ldots, B_n, A_1) \\ \Vert\, \mathsf{KS}\uparrow \\ \mathsf{false} \end{array}.$$

Build $\boxed{\begin{array}{c} \mathsf{w}\uparrow \dfrac{(B_1, \ldots, B_n, (A_1, A_2))}{(B_1 \ldots B_n, A_1)} \\ \Vert\, \mathsf{KS}\uparrow \\ \mathsf{false} \end{array}}$. Contradiction. $\square$

Lemma 4 $\quad [A_1, A_2] \in \Gamma^* \Rightarrow A_1 \in \Gamma^*$ or $A_2 \in \Gamma^*$

Proof: Assume $[A_1, A_2] \in \Gamma^*$ and $A_1 \notin \Gamma^*$ and $A_2 \notin \Gamma^*$.
Then we have $\{B_1 \ldots B_n\} \subseteq \Gamma^*$ and $\{C_1 \ldots C_m\} \subseteq \Gamma^*$

with $\begin{array}{c} (B_1 \ldots B_n, A_1) \\ \Delta_1 \Vert\, \mathsf{KS}\uparrow \\ \mathsf{false} \end{array}$ and $\begin{array}{c} (\{C_1 \ldots C_m\}, A_2) \\ \Delta_2 \Vert\, \mathsf{KS}\uparrow \\ \mathsf{false} \end{array}$

subset of Γ^*

Build

$$([A_1, A_2],\ \text{some } B\text{'s},\ \text{some } C\text{'s})$$

$$\Big\| \; c\uparrow \quad \text{— to remove duplicates ct } \frac{B_i}{\underbrace{(B_i, C_j)}_{=}}$$

$$s^2 \; \frac{([A_1, A_2],\ B_1 \dots B_n,\ C_1 \dots C_m)}{[(B_1 \dots B_n, A_1),\ (C_1 \dots C_m, A_2)]}$$

$$\Delta_1 \Big\| \qquad\qquad \Delta_2 \Big\|$$

$$= \; \frac{[\ \text{false} \quad , \quad \text{false}\ ]}{\text{false}}$$

Contradiction. $\square$

Modal Logic

Definition System SKSk is system SKS extended with:

Necessitation: $\Box\,\mathrm{true} = \mathrm{true}$

$\Diamond\,\mathrm{false} = \mathrm{false}$

$$k{\downarrow}\ \frac{\Box[A,B]}{[\Box A, \Diamond B]} \qquad\qquad k{\uparrow}\ \frac{(\Box A, \Diamond B)}{\Diamond (A,B)}$$

$$\cancel{j{\downarrow}\ \frac{[\Diamond A, \Diamond B]}{\Diamond [A,B]} \qquad\qquad j{\uparrow}\ \frac{\Box (A,B)}{(\Box A, \Box B)}}$$

– DERIVABLE!

– Derivable

Derivable

KSk↓ and KSk↑ are defined analogous to SKS.

Theorem KSk↓ is complete.

Proof analogous to the one for KS↓, implication (2) is the next theorem.

Theorem If a formula F is KSK↑-consistent, then it is satisfiable.

Proof Γ^* is the max. cons. extension of $\{F\}$, which exist by Lemma 1 (with KS↑ replaced by KSK↑).

Define $M = (S, \pi : S \times V \to \{true, false\}, K)$ as

$$S = \{ s_V \mid V \text{ is a max. cons. set} \}$$

$$\pi(s_V, p) = \begin{cases} \text{true} & \text{if } p \in V \\ \text{false} & \text{otherwise} \end{cases}$$

$$K = \{ (s_V, s_W) \mid V/\boxtimes \subseteq W \}$$

Claim: $A \in V$, V max. cons. $\Rightarrow (M, s_V) \models A$

by ind. on A.

for $A = a, \bar{a}, (A_1, A_2), [A_1, A_2]$ see prop. case.

$A = \Box A'$: $\Box A' \in V \Rightarrow A' \in V/\Box \Rightarrow$

$(\forall s_W . (s_V, s_W) \in K \Rightarrow A' \in W) \underset{IH}{\Rightarrow}$

$(\forall s_W . (s_V, s_W) \in K \Rightarrow (M, s_W) \models A')) \underset{def}{\Rightarrow} (M, s_V) \models \Box A'$

$A = \Diamond A'$: $\Diamond A' \in V \underset{\text{Lemma 5}}{\Rightarrow}$ there is a max. cons. set W, st.

$A' \in W$ and $W \supseteq V/\Box \underset{IH + Def. V/\Box}{\Longrightarrow} (M, s_W) \models A'$

and $(s_V, s_W) \in K \Rightarrow (M, s_V) \models \Diamond_i A'$.

$\Box$

Lemma 5 Given a max. cons. set V with $\Diamond A' \in V$, then is a max. cons. set W st.

$$V/\Box \cup \{A'\} \subseteq W.$$

Proof By Lemma 1 we just need to show that

$V/\Box \cup \{A'\}$ is consistent. Assume it's not.

then we have for a subset $\{F_1 \dots F_n\}$

$$\begin{array}{c} (F_1, \dots, F_n, A') \\ \Vert \; \mathsf{KS}_{\mathsf{K}\uparrow} \\ \mathsf{false} \end{array}.$$

Construct

$$\begin{array}{lc} & (\Box F_1 \dots \Box F_n, \Diamond A') \quad \leftarrow \text{subset of } V \\ \mathsf{K}\uparrow^n \; (\text{+ assoc}) & \overline{} \\ & \Diamond(F_1 \dots F_n, A') \\ & \Vert \\ = & \dfrac{\Diamond\, \mathsf{false}}{\mathsf{false}} \end{array}.$$

Contradiction. $\Box$