Finitisation for Propositional Linear Time Logic

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Abstract. Currently known sequent systems for propositional linear time temporal logic either include a cut rule in some form or an *infinitary* rule, which is a rule with infinitely many premises. We strengthen the infinitary rule to require only a finite number of premises and show that this modification preserves soundness. This way we obtain a finitary cut-free sequent system which almost fulfills the subformula property. However, the number of premises of the finitary rule is exponential in the size of its conclusion and the soundness proof relies on the small model property.

1 Introduction

Linear time temporal logics are used for specification and verification of reactive systems. A recent overview of the development of this subject and a collection of references can be found in Lichtenstein and Pnueli [9]. Several approaches to the design of deductive systems and decision procedures have been proposed for this logic. The paper just mentioned gives a tableau-based decision procedure as well as a Hilbert-style axiom system and a proof of its completeness. A resolution-based approach is given by Fisher et al. in [1], an approach based on a translation into Büchi automata is given by Sistla et al. in [13], and a game-theoretic approach can be found in Lange and Stirling in [8].

In this paper, we study cut-free sequent systems for propositional linear temporal logic. A cut-free sequent system is a valuable tool that helps us understand a logic. It is typically much easier to carry out proof search in a cut-free sequent system than in a Hilbert-style axiom system. Proof search in the sequent calculus is typically easy to understand because of the clear logical reading of the inference rules. We feel that the same cannot be said, for example, for a procedure that computes strongly connected components in a graph, which is part of tableau procedures.

Several sequent systems have already been designed for linear temporal logic, but they all have their problems. Either they are *infinitary*, i.e. they contain a rule with an infinite number of premises such as

$$\omega \, \frac{\vdash \Gamma, A \quad \vdash \Gamma, \bigcirc A \quad \vdash \Gamma, \bigcirc\bigcirc A \quad \dots}{\vdash \Gamma, \Box A} \qquad ,$$

or they are not truly cut-free, that is they contain a rule such as

$$\Box \frac{\vdash \Gamma, B \quad \vdash \bar{B}, \bigcirc B \quad \vdash \bar{B}, A}{\vdash \Gamma, \Box A} \quad ,$$

which clearly violates the subformula property since B is an arbitrary formula. Kawai [7] and Szałas [14] give systems of the first kind. Gudzhinskas [4], Paech [10] and Szałas [15] give systems of the second kind. An exception is Pliuškevičius [11], who gives a finitary and truly cut-free system for a fragment of linear temporal logic with first-order quantifiers. However, this fragment does not include full propositional linear temporal logic.

Typically, a tableau system closely corresponds to a cut-free sequent system. Unfortunately this is not true in the case of temporal logics, cf. [3]. Here the tableau procedure consists of two passes: in the first it constructs a certain graph, from which it deletes certain strongly connected components in the second pass. This two-pass nature gets in the way of a correspondence to a sequent system. Schwendimann gives a one-pass tableau procedure in [12], but it works on a more intricate data structure than sets of formulas and thus here as well it is hard to see a correspondence to a sequent system.

Jäger et al. propose using an ω-rule with finitely many premises for the logic of common knowledge and for a fragment of the modal μ-calculus in [5,6]. Here we adapt this idea in order to obtain a finitary cut-free sequent system for linear temporal logic. In the following section we present syntax and semantics of unary linear temporal logic and also give the proof of the standard canonical model theorem. In Section 3 we show an infinitary sequent system and prove completeness of this system by establishing a connection with the tableau graph. In Section 4, which is the heart of this paper, we give the finitary sequent system and show that it is equivalent to the infinitary one. In Section 5 we extend this result to full linear time temporal logic, i.e. we add the connectives *until* and *release*. Some discussion in Section 6 concludes this paper.

2 Preliminaries

Propositions p and their negations $\bar{p}$ are *atoms*, with $\bar{\bar{p}}$ defined to be p. Atoms are denoted by a, b, c and so on. The *formulas* of unary LTL, denoted by A, B, C, D are given by the grammar

$$A ::= \bigcirc^n a \mid (A \vee A) \mid (A \wedge A) \mid \Diamond A \mid \Box A \quad ,$$

where $\bigcirc^n$ denotes a sequence of n connectives $\bigcirc$ for some $n \geq 0$. A formula of the form $\bigcirc^n a$ is called *elementary*, and its intended meaning is that a holds n steps from now. The intended meaning of $\Diamond A$ is "eventually A will hold" and that of $\Box A$ is "A will always hold".

Given a formula A, its *negation* $\bar{A}$ is defined as follows:

$$\overline{\bigcirc^n a} = \bigcirc^n \bar{a} \qquad \begin{array}{l} \overline{A \vee B} = \bar{A} \wedge \bar{B} \\ \overline{A \wedge B} = \bar{A} \vee \bar{B} \end{array} \qquad \begin{array}{l} \overline{\Diamond A} = \Box \bar{A} \\ \overline{\Box A} = \Diamond \bar{A} \end{array} \qquad ,$$

and the *next operator* applied on A, denoted $\bigcirc A$, is defined as follows:

$$\begin{array}{ll} \bigcirc(A \vee B) = \bigcirc A \vee \bigcirc B & \quad \bigcirc \Diamond A = \Diamond \bigcirc A \\ \bigcirc(A \wedge B) = \bigcirc A \wedge \bigcirc B & \quad \bigcirc \Box A = \Box \bigcirc A \end{array} \qquad .$$

A *subformula* of a formula A is a substring of A that is a formula. In particular, $\bigcirc^n a$ is a subformula of $\bigcirc^m a$ if $n \leq m$. The *size* of a formula A is the number of its subformulas and is denoted by $|A|$.

A *model*, denoted by σ, is an ω-sequence of sets of propositions. The element of the sequence at position i is denoted by $\sigma(i)$ and $\sigma(0)$ denotes the first element.

We define the relation $\models$ as follows:

$$\begin{array}{l} \sigma, i \models \bigcirc^n p \iff p \in \sigma(i+n) \\ \sigma, i \models \bigcirc^n \bar{p} \iff p \notin \sigma(i+n) \\ \sigma, i \models A \vee B \iff \sigma, i \models A \text{ or } \sigma, i \models B \\ \sigma, i \models A \wedge B \iff \sigma, i \models A \text{ and } \sigma, i \models B \\ \sigma, i \models \Diamond A \iff \exists j \geq i \ \sigma, j \models A \\ \sigma, i \models \Box A \iff \forall j \geq i \ \sigma, j \models A \\ \sigma \models A \iff \forall i \ \sigma, i \models A \qquad . \end{array}$$

A formula A is *valid*, denoted by $\models A$, if $\forall \sigma \forall i \ \sigma, i \models A$ and it is *satisfiable* if $\exists \sigma \exists i \ \sigma, i \models A$. The following proposition easily follows from the above definition:

Proposition 1. *For all models σ, all $i \geq 0, n \geq 0$ and all formulas A we have the following equivalences:*

$$\begin{array}{l} \sigma, i \models A \iff \sigma, i \not\models \bar{A} \\ \sigma, i \models \bigcirc^n A \iff \sigma, i+n \models A \qquad . \end{array}$$

The *closure* of A, denoted by $cl(A)$, is the set of all subformulas of A and their negations. A subset s of $cl(A)$ is *saturated* if the following conditions are fulfilled:

$$\begin{array}{l} \forall \bigcirc^n a \in cl(A) \quad \bigcirc^n a \in s \implies \bigcirc^n \bar{a} \notin s \\ \text{if } B \vee C \in s \text{ then } B \in s \text{ or } C \in s \\ \text{if } B \wedge C \in s \text{ then } B \in s \text{ and } C \in s \\ \text{if } \Box B \in s \text{ then } B \in s \qquad . \end{array}$$

The *tableau graph* of A, denoted by $G(A)$, has all saturated subsets of $cl(A)$ as its nodes and there is an edge from a node s to a node t if the following conditions are fulfilled, where $n \geq 0$:

$$\begin{array}{l} \text{if } \bigcirc^{n+1} a \in s \text{ then } \bigcirc^n a \in t \\ \text{if } \Box B \in s \text{ then } \Box B \in t \qquad . \end{array}$$

A *path* through a graph is a sequence of nodes in the graph such that for two consecutive nodes in the sequence there is an edge from the first to the second. Paths can be empty and paths can be infinite. A *fulfilling path* is a path π through a tableau graph such that for all $i \geq 0$ and for all formulas B we have that

$$\text{if } \Diamond B \in \pi(i) \text{ then } \exists j \geq i \; B \in \pi(j) \qquad .$$

If a path is fulfilling and starts with a node that contains the formula A then it is *fulfilling for* A. An infinite path π through a tableau graph induces a model $\tilde{\pi}$, which is obtained by dropping all formulas that are not propositions.

The *size* of a tableau graph $G(A)$, denoted by $|G(A)|$ is its number of nodes. Since for any formula A we have $|cl(A)| \leq 2 \cdot |A|$ we also have $|G(A)| \leq 2^{2 \cdot |A|}$.

The main reason for defining the tableau graph is the canonical model theorem for linear time logic. We adapt it from [9]. The theorem says that instead of looking for arbitrary models for a formula, we can restrict ourselves to paths in the finite tableau graph.

Theorem 2.

1. *A formula of unary LTL is satisfiable iff there is an infinite fulfilling path for it in its tableau graph.*
2. *Let π be an infinite fulfilling path in the tableau graph of A. Then for each formula $B \in cl(A)$ and each $i \geq 0$ we have that $B \in \pi(i) \implies \tilde{\pi}, i \models B$.*

Proof. To prove the first statement left-to-right, consider a formula A, a model σ and a natural number i such that $\sigma, i \models A$. Define a sequence π as $\pi(j) = \{B \in cl(A) \mid \sigma, i + j \models B\}$. It is easy to check that each $\pi(j)$ is saturated and that π is an infinite fulfilling path for A. The right-to-left direction of the first statement follows from the second statement. We prove it by induction on B. Let B be an elementary formula $\bigcirc^n p$. Then we have for all $i \geq 0$ the following implications, which easily follow from the definitions:

$$\bigcirc^n p \in \pi(i) \implies p \in \pi(i+n) \iff p \in \tilde{\pi}(i+n) \iff \tilde{\pi}, i \models \bigcirc^n p \quad .$$

The case of an elementary formula $\bigcirc^n \bar{p}$ works similarly. The cases of the propositional connectives are standard. Let $B = \Diamond C$. We have for all $i \geq 0$ that

$$\Diamond C \in \pi(i) \implies \exists j \geq i \; C \in \pi(j) \implies \exists j \geq i \; \tilde{\pi}, j \models C \iff \tilde{\pi}, i \models \Diamond C \quad ,$$

where the left implication is due to π being a fulfilling path and the middle implication is by induction hypothesis. Let $B = \Box C$. We have for all $i \geq 0$ that

$$\Box C \in \pi(i) \implies \forall j \geq i \; C \in \pi(j) \implies \forall j \geq i \; \tilde{\pi}, j \models C \iff \tilde{\pi}, i \models \Box C \quad ,$$

where the left implication is due to saturation of the elements of π and the definition of the tableau graph and the middle implication is by induction hypothesis. □

$$\vdash \Gamma, \bigcirc^n a, \bigcirc^n \bar{a} \qquad \wedge\,\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \qquad \vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\Box^\omega\,\frac{\vdash \Gamma, \bigcirc^n A \quad \text{for all } n \geq 0}{\vdash \Gamma, \Box A} \qquad \Diamond\,\frac{\vdash \Gamma, \bigcirc\Diamond A, A}{\vdash \Gamma, \Diamond A}$$

Fig. 1. An infinitary sequent system LT^ω

3 An Infinitary Sequent System

We use Γ to denote a finite multiset of formulas. An infinitary sequent system for unary linear temporal logic is shown in Figure 1.

According to our notion of subformula this system does not fulfill the subformula property. The formula $\bigcirc p$, for example, is not a subformula of $\Diamond p$. However, the same is true for Gentzen's system LK [2], where $A[x/t]$ generally is not a subformula of $\exists x A$. Our point here is that we have a subformula property modulo the next connective, which is much better than introducing arbitrary formulas in the premise.

Soundness and completeness of essentially the same sequent system (but including quantifiers) have also been proved e.g. in Pliuškevičius [11]. Our proof is different in making use of Theorem 2 and thus establishing a connection between branches in proofs in the sequent system and paths in the tableau graph.

The *corresponding formula* of a finite multiset of formulas is the disjunction of all formulas in its underlying set. When no confusion arises, we write Γ both for the multiset and for its corresponding formula.

Theorem 3. *System* LT^ω *is sound and complete.*

Proof. Soundness follows from an easy induction on the proof. To show completeness we devise a proof search strategy that allows to construct a countermodel from a failed proof search. We build a possibly infinite-depth derivation for the sequent $\vdash \Gamma$ in a root-first fashion: 1) choose an unmarked non-elementary formula in some non-axiomatic leaf and apply the corresponding rule to it. 2) Mark the formulas produced by the inference rule if they are new. Repeat steps 1) and 2) until each non-elementary formula in each non-axiomatic leaf is marked. 3) Remove all markings. Repeat the steps above until a fixpoint is reached. If this process terminates with all leaves axiomatic, then we have a proof. So assume that this process either terminates with a non-axiomatic leaf or does not terminate. Let Γ^* be the union of all sequents on the problematic (i.e. non-axiomatic or infinite) branch. Define a sequence π of sets of formulas such that $\pi(i) = \{A \in cl(\Gamma) \mid \bigcirc^i \bar{A} \in \Gamma^*\}$. It is easy to prove that π is a fulfilling path in $G(\Gamma)$. The saturation of each $\pi(i)$ follows from the axiom, the rules for $\wedge, \vee$ and

$\Diamond$. The fact that there is an edge from $\pi(i)$ to $\pi(i+1)$ follows by definition of π and the $\Diamond$-rule, and fulfillingness of π follows from the $\Box^\omega$-rule. In case π is finite it is of the form $\pi' s$ where no formula $\bigcirc B \in s$ and thus there is an edge from s to itself in $G(\Gamma)$. We thus have an infinite fulfilling path $\pi' sssss\ldots$.

Since for each $A \in \Gamma$ we have $\bar{A} \in \pi(0)$, by Theorem 2 we have a countermodel for Γ. Since thus a failure of proof search yields a countermodel for Γ proof search succeeds for a valid Γ. $\Box$

4 A Finitary Sequent System

System $\mathsf{LT}^{<\omega}$ is obtained from system LT^{ω} by replacing the $\Box^\omega$-rule by the following finitary rule:

$$\Box^{<\omega}\ \frac{\vdash \Gamma, \bigcirc^n A \quad \text{for all } n \le 2^{4\cdot|\Gamma \vee \Box A|}}{\vdash \Gamma, \Box A} \quad .$$

Since this rule allows to derive the same conclusion as the $\Box^\omega$-rule but with fewer premises, it is not clear that this rule is sound. To see this, we first need a few definitions.

Definition 4. A path of the form $s\pi s$, where s is a node and π is a path, is a *cycle*. We say that the path π is *properly contained* in the cycle $s\pi s$. A cycle $s\pi s$ in a path $\pi_1 s\pi s\pi_2$ is *redundant* if each node in π occurs in π_2. A path is *reduced* if it does not contain redundant cycles.

Example 5. Consider the fully connected graph formed by the three nodes s, t, u. Then the path stu is reduced: it contains no cycle at all. The path $stsu$ is reduced because its only cycle sts is not redundant. In contrast, both paths ssu and $stsut$ are not reduced because the first contains the redundant cycle ss and the second contains the redundant cycle sts.

The notion of redundancy is designed to give us two things: first, removing redundant cycles preserves the property of being fulfilling:

Lemma 6. *If the cycle $s\pi s$ is redundant in the fulfilling path $\pi_1 s\pi s\pi_2$ then $\pi_1 s\pi_2$ is a fulfilling path.*

Second, it gives us an upper bound on the size of reduced paths:

Lemma 7. *In a graph of size n the length of each reduced path is smaller than or equal to $\sum_{i=0}^{n} i$.*

Proof. We proceed by induction on n, the case $n = 0$ is obvious. For the induction step we argue by contradiction: assume that we have a reduced path π with

$|\pi| > \sum_{i=0}^{n+1} i$ in a graph of size $n+1$. Then $\pi = \pi_1 s \pi_2$ with $|\pi_1| = n+1$ and $|s\pi_2| > \sum_{i=0}^{n} i$. Because of its length, $\pi_1 s$ contains a cycle and by reducedness this cycle properly contains at least one node. Again because of reducedness, $s\pi_2$ does not contain this node, so it is a reduced path in a graph of size n, which contradicts the induction hypothesis. $\square$

Theorem 8. *The $\Box^{<\omega}$-rule is sound.*

Proof. Assuming the validity of all the premises of the $\Box^{<\omega}$-rule we prove the validity of its conclusion. We have $\models \Gamma \vee \Box A \iff \bar{\Gamma} \wedge \Diamond \bar{A}$ is unsatisfiable $\iff$ there is no infinite fulfilling path for $\bar{\Gamma} \wedge \Diamond \bar{A}$ in $G(\bar{\Gamma} \wedge \Diamond \bar{A})$.

Assume for the sake of contradiction that there is such an infinite fulfilling path. Then it is of the form

$$\pi \quad = \quad s\,\pi_1\,t\,\pi_2 \quad \text{, with } \bar{\Gamma} \wedge \Diamond \bar{A} \in s \text{ and } \bar{A} \in t \quad .$$

Using Lemma 6, we remove all redundant cycles in $s\pi_1 t$ to obtain a fulfilling path $s\pi_1' t$. Let i be the position of t in that path. By Lemma 7 we have that $i \leq 2^{4\cdot|\Gamma \vee \Box A|}$. Let σ be the model induced by the path $s\,\pi_1'\,t\,\pi_2$. By Theorem 2 we have that $\sigma, 0 \models \bar{\Gamma}$ and that $\sigma, i \models \bar{A}$. Thus $\sigma \not\models \Gamma \vee \bigcirc^i A$, which contradicts the validity of some premise of the $\Box^{<\omega}$-rule. $\square$

Theorem 9. *System LT^{ω} and system $\mathsf{LT}^{<\omega}$ are equivalent.*

Proof. Clearly, given a proof of a sequent in LT^{ω} we get a proof of the same sequent in $\mathsf{LT}^{<\omega}$ simply by removing branches from instances of the $\Box^{\omega}$-rule. The other direction follows from completeness of LT^{ω} and soundness of $\mathsf{LT}^{<\omega}$, which follows from Theorem 8. $\square$

5 Until and Release

To consider full linear temporal logic we define *formulas* of the language LTL by the grammar

$$A ::= \bigcirc^n a \mid (A \vee A) \mid (A \wedge A) \mid (A\,\mathcal{U}\,A) \mid (A\,\mathcal{R}\,A) \quad ,$$

where $\mathcal{U}$ is the connective *until* and $\mathcal{R}$ is the connective *release*. We replace the definitions of negation and next for the connectives $\Diamond$ and $\Box$ as follows:

$$\begin{array}{ll} \overline{A\,\mathcal{U}\,B} = \bar{A}\,\mathcal{R}\,\bar{B} & \qquad \bigcirc(A\,\mathcal{U}\,B) = (\bigcirc A)\,\mathcal{U}\,(\bigcirc B) \\ \overline{A\,\mathcal{R}\,B} = \bar{A}\,\mathcal{U}\,\bar{B} & \qquad \bigcirc(A\,\mathcal{R}\,B) = (\bigcirc A)\,\mathcal{R}\,(\bigcirc B) \end{array} \quad ,$$

and the definitions of the relation $\models$ as follows:

$$\begin{array}{l} \sigma, i \models A\,\mathcal{U}\,B \iff \exists j \geq i \ \ (\sigma, j \models B \text{ and } \forall i \leq k < j \ \ \sigma, k \models A) \\ \sigma, i \models A\,\mathcal{R}\,B \iff \forall j \geq i \ \ (\sigma, j \models B \ \text{ or } \ \exists i \leq k < j \ \ \sigma, k \models A) \end{array} \quad .$$

The unary connectives $\Diamond$ and $\Box$ are definable using the binary ones. It is easy to check that $\Diamond A$ is equivalent to $\top \,\mathcal{U}\, A$, where $\top = a \vee \bar{a}$ for some atom a, and dually so for the connective $\Box$.

The condition for the connective $\Box$ in the definition of a saturated subset is replaced by

if $B \,\mathcal{U}\, C \in s$ then $B \in s$ or $C \in s$
if $B \,\mathcal{R}\, C \in s$ then $C \in s$,

and in the definition of the tableau graph the condition is replaced by

if $B \,\mathcal{U}\, C \in s$ and $C \notin s$ then $B \,\mathcal{U}\, C \in t$
if $B \,\mathcal{R}\, C \in s$ and $B \notin s$ then $B \,\mathcal{R}\, C \in t$.

A *fulfilling path* is a path π through $G(A)$ such that for all $i \geq 0$ and for all formulas $B \,\mathcal{U}\, C$ we have that

if $B \,\mathcal{U}\, C \in \pi(i)$ then $\exists j \geq i \; C \in \pi(j)$.

Theorem 10.

1. *A formula of LTL is satisfiable iff there is an infinite fulfilling path for it in its tableau graph.*
2. *Let π be an infinite fulfilling path in the tableau graph of A. Then for each formula $B \in cl(A)$ and each $i \geq 0$ we have that $B \in \pi(i) \implies \tilde{\pi}, i \models B$.*

Proof. We copy the proof of Theorem 2 except for the two inductive cases for $\Diamond$ and $\Box$, which are replaced as follows:

Let $B = C \,\mathcal{U}\, D$. We have for all $i \geq 0$ that

$$\begin{aligned} C \,\mathcal{U}\, D \in \pi(i) &\implies \exists j \geq i \; (D \in \pi(j) \;\wedge\; \forall i \leq k < j \; C \in \pi(k)) \\ &\implies \exists j \geq i \; (\tilde{\pi}, j \models D \;\wedge\; \forall i \leq k < j \; \tilde{\pi}, k \models C) \\ &\iff \tilde{\pi}, i \models C \,\mathcal{U}\, D \quad , \end{aligned}$$

for the first implication the existence of a j that satisfies the first conjunct follows from fulfillingness of the path π. Take the least j that satisfies the first conjunct. Then for all $\pi(k)$ with $i \leq k < j$ we have $D \notin \pi(k)$. Then by definition of the tableau graph $C \,\mathcal{U}\, D \in \pi(k)$ and by saturation $C \in \pi(k)$. Thus the second conjunct is fulfilled. The second implication is by induction hypothesis.

Let $B = C \,\mathcal{R}\, D$. We have for all $i \geq 0$ that

$$\begin{aligned} C \,\mathcal{R}\, D \in \pi(i) &\implies \forall j \geq i \; (D \in \pi(j) \;\vee\; \exists i \leq k < j \; C \in \pi(k)) \\ &\implies \forall j \geq i \; (\tilde{\pi}, j \models D \;\vee\; \exists i \leq k < j \; \tilde{\pi}, k \models C) \\ &\iff \tilde{\pi}, i \models C \,\mathcal{R}\, D \quad , \end{aligned}$$

where the second implication is by induction hypothesis. For the first implication, take some j with $j \geq i$ that does not satisfy the second disjunct. Then for all $\pi(k)$ with $i \leq k < j$ by definition of the tableau graph we have $C \,\mathcal{R}\, D \in \pi(k)$ and also $C \,\mathcal{R}\, D \in \pi(j)$. By saturation we have $D \in \pi(j)$. Thus all such j fulfill the first disjunct. $\Box$

$$\vdash \Gamma, \bigcirc^n a, \bigcirc^n \bar{a} \qquad \wedge\,\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \qquad \vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

$$\mathcal{R}^\omega\,\frac{\vdash \Gamma, B \quad \vdash \Gamma, \bigcirc(A\,\mathcal{R}\,B), \bigcirc^0 A, \ldots, \bigcirc^{n-1} A, \bigcirc^n B \quad \text{for all } n \geq 1}{\vdash \Gamma, A\,\mathcal{R}\,B}$$

$$\mathcal{U}\,\frac{\vdash \Gamma, A, B \qquad \vdash \Gamma, \bigcirc(A\,\mathcal{U}\,B), B}{\vdash \Gamma, A\,\mathcal{U}\,B}$$

Fig. 2. An infinitary sequent system $\mathsf{LT}^\omega_{\mathcal{U},\mathcal{R}}$

An infinitary sequent system for full LTL is shown in Figure 2.

Theorem 11. *System $\mathsf{LT}^\omega_{\mathcal{U},\mathcal{R}}$ is sound and complete.*

Proof. The proof works like that of Theorem 3. □

We now obtain a finitary system $\mathsf{LT}^{<\omega}_{\mathcal{U},\mathcal{R}}$ from system $\mathsf{LT}^\omega_{\mathcal{U},\mathcal{R}}$ by replacing the $\mathcal{R}^\omega$-rule by the following finitary rule:

$$\mathcal{R}^\omega\,\frac{\vdash \Gamma, B \quad \vdash \Gamma, \bigcirc(A\,\mathcal{R}\,B), \bigcirc^0 A, \ldots, \bigcirc^{n-1} A, \bigcirc^n B \quad \text{for all } n \leq 2^{4\cdot|\Gamma \vee A\mathcal{R}B|}}{\vdash \Gamma, A\,\mathcal{R}\,B} \quad .$$

Just as in the unary case, we have to prove that this rule is sound:

Theorem 12. *The $\mathcal{R}^{<\omega}$-rule is sound.*

Proof. Assuming the validity of all the premises of the $\mathcal{R}^{<\omega}$-rule we prove the validity of its conclusion. We have $\models \Gamma \vee A\,\mathcal{R}\,B \iff \bar{\Gamma} \wedge \bar{A}\,\mathcal{U}\,\bar{B}$ is unsatisfiable $\iff$ there is no infinite fulfilling path for $\bar{\Gamma} \wedge \bar{A}\,\mathcal{U}\,\bar{B}$ in its tableau graph.

Assume for the sake of contradiction that there is such an infinite fulfilling path. Then either $\bar{B}$ is in the first state of this path, which contradicts the validity of the left-most premise, or the path is of the form

$$\pi \quad = \quad s\,\pi_1\,t\,\pi_2 \quad , \text{with } \bar{\Gamma} \wedge \bar{A}\,\mathcal{U}\,\bar{B} \in s \text{ and } \bar{B} \in t \quad ,$$

such that $\bar{A}$ is in s and in every node of π_1. Using Lemma 6, we remove all redundant cycles in $s\pi_1 t$ to obtain a fulfilling path $s\pi_1' t$. Let i be the position of t in that path. By Lemma 7 we have that $i \leq 2^{4\cdot|\Gamma \vee A\mathcal{R}B|}$. Let σ be the model induced by the path $s\,\pi_1'\,t\,\pi_2$. By Theorem 10 we have that $\sigma, 0 \models \bar{\Gamma}$, $\forall j < i\ \sigma, j \models \bar{A}$ and that $\sigma, i \models \bar{B}$. Thus $\sigma \not\models \Gamma \vee \bigcirc^0 A \vee \cdots \vee \bigcirc^{i-1} A \vee \bigcirc^i B$, which contradicts the validity of some premise of the $\mathcal{R}^{<\omega}$-rule. □

Just like in the unary case we have equivalence of the infinitary and the finitary system:

Theorem 13. *System* $\mathsf{LT}^{\omega}_{\mathcal{U},\mathcal{R}}$ *and system* $\mathsf{LT}^{<\omega}_{\mathcal{U},\mathcal{R}}$ *are equivalent.*

6 Discussion

We have given sequent systems for unary and for full linear temporal logic, which are finitary, cut-free and almost satisfy the subformula property. To the best of our knowledge, these are the first sequent systems for these logics that satisfy these properties.

There are two problems that we are currently addressing. The first is obtaining a decision procedure based on the given finitary sequent system. Clearly the proof search procedure given in the completeness proof immediately gives us a semi-decision procedure once we replace the infinitary rule by the finitary rule. It should be possible to extract a countermodel if after some finite number of steps the proof search has not succeeded.

The second problem is obtaining better bounds on the number of premises of the $\Box^{<\omega}, \mathcal{R}^{<\omega}$-rules. While we do not see a way to prevent the exponential, it seems possible to define a bound in terms of the endsequent of the proof rather than in terms of the conclusion of the rule.

After that, of course we have to set the bar higher than just asking for finitariness and the subformula property. Clearly, the $\Box^{<\omega}, \mathcal{R}^{<\omega}$-rules are problematic: the exponential number of premises probably spoils uses in interactive proof search. A sequent system with a constant small number of premises for each rule would be desirable, and should be a target of future research. Further problems, that seem challenging and worthwhile to us, are obtaining cut-free systems for more expressive temporal logics, such as the linear μ-calculus or (larger fragments of) temporal logic with first-order quantifiers. Obtaining a syntactic cut elimination procedure is also an interesting direction for future research.

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