Modular Sequent Systems for Modal Logic

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Abstract. We see cut-free sequent systems for the basic normal modal logics formed by any combination the axioms $d, t, b, 4, 5$. These systems are *modular* in the sense that each axiom has a corresponding rule and each combination of these rules is complete for the corresponding frame conditions. The systems are based on *nested sequents*, a natural generalisation of hypersequents. Nested sequents stay inside the modal language, as opposed to both the display calculus and labelled sequents. The completeness proof is via syntactic cut elimination.

1 Introduction

This paper is part of a research effort to develop proof-theoretic systems for modal logic that stay inside the modal language. This requirement in particular excludes the display calculus [2, 16], which is formulated in the language of tense logic, and labelled sequents, see for example [12], which are formulated in a hybrid language.

Examples of modal proof systems that stay inside the modal language are hypersequent systems [1], systems in the calculus of structures [10, 14, 15, 9] and, what we will study here, systems using *nested sequents*. Nested sequents are a natural generalisation of hypersequents, and they have been invented several times independently, for example by Bull [6], by Kashima [11], by Brünnler [4, 3] (using the name *deep sequent* at the time) and by Poggiolesi (using the name *tree-hypersequent*) [13].

In this paper we provide cut-free sequent systems for the basic normal modal logics formed from the axioms $d, t, b, 4, 5$ which are *modular* in the sense that each modal axiom has a corresponding sequent calculus rule and that each combination of these rules is sound and complete for the corresponding modal logic. To our knowledge, these are the first systems inside the modal language which are modular.

These systems are closely related to the systems introduced in [3], in particular the subsystem for the modal logic K is essentially the same. Modal axioms in [3] were turned into $\diamond$-rules, that is, rules where the active formula in the conclusion has the connective $\diamond$ as its main connective. However, these systems are

not modular. For example, the logic S5 can be axiomatised in a Hilbert system both by using the axioms t, b, 4 and by using t, 5. While there is a complete cut-free system for S5 in [3], namely one with the rules t, b, 4, 5, neither the cut-free system with rules t, b, 4 nor the cut-free system with the rules t, 5 are complete for S5. Here, we follow another approach to designing modal rules. We do not turn modal axioms into $\Diamond$-rules, but into structural rules. These structural rules are the key to modularity, as was already conjectured in [4].

Proving completeness for our systems turned out to be challenging. We do so by way of embedding a corresponding Hilbert system and syntactically proving cut-elimination. The cut-elimination proof is interesting: it relies on a decomposition of the contraction rule, similar to what has been observed in deep inference systems for propositional logic, where contraction is decomposed into an atomic version and a local *medial* rule [5].

2 The Sequent Systems

Formulas and modal axioms. Propositions p and their negations $\bar{p}$ are *atoms*, with $\bar{\bar{p}}$ defined to be p. *Formulas*, denoted by A, B, C, D are given by the grammar

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond A \mid \Box A \quad .$$

Given a formula A, its *negation* $\bar{A}$ is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\bar{A} \vee B$ and $\top$ and $\bot$ are respectively defined as $p \vee \bar{p}$ and $p \wedge \bar{p}$ for some proposition p. Each name in $\{k, d, t, b, 4, 5\}$ corresponds both to a frame condition and to a Hilbert-style axiom:

k:	$\top$	$\Box(A \vee B) \supset (\Box A \vee \Diamond B)$	b: symmetric	$A \supset \Box \Diamond A$
d:	serial	$\Box A \supset \Diamond A$	4: transitive	$\Box A \supset \Box \Box A$
t:	reflexive	$A \supset \Diamond A$	5: euclidean	$\Diamond A \supset \Box \Diamond A$

Nested sequents. A *nested sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]$ where Γ is a nested sequent. In the following, a *sequent* is a nested sequent. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi, \Sigma$. As usual, sequents are written without curly braces and the comma in the expression Γ, Δ is multiset union. A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1], \ldots, [\Delta_n] \quad .$$

The *corresponding formula* of the above sequent is $\bot$ if $m = n = 0$ and otherwise

$$A_1 \vee \cdots \vee A_m \vee \Box(D_1) \vee \cdots \vee \Box(D_n) \quad ,$$

where $D_1 \ldots D_n$ are the corresponding formulas of the sequents $\Delta_1 \ldots \Delta_n$. Notice that a sequent induces a tree where each node is marked with a multiset of formulas. We will refer to notions such as the *depth* or the *root* of a sequent, meaning the depth or the root of the corresponding tree.

Sequent contexts. Informally, a context is a sequent with holes. We will mostly encounter sequents with just one or two holes. A *unary context* is a

2

sequent with exactly one occurrence of the symbol { }, the *hole*, which does not occur inside formulas. Such contexts are denoted by $\Gamma\{\ \}$, $\Delta\{\ \}$, and so on. The hole is also called the *empty context*. The sequent $\Gamma\{\Delta\}$ is obtained by replacing { } inside $\Gamma\{\ \}$ by Δ. For example, if $\Gamma\{\ \} = A, [[B], \{\ \}]$ and $\Delta = C, [D]$ then

$$\Gamma\{\Delta\} = A, [[B], C, [D]] \quad .$$

The *depth* of a unary context $\Gamma\{\ \}$, denoted $depth(\Gamma\{\ \})$ is defined as follows

$$depth(\Gamma, \{\ \}) = 0$$
$$depth(\Gamma, [\Delta\{\ \}]) = depth(\Delta\{\ \}) + 1.$$

More generally, a *context* is a sequent with $n \geq 0$ occurrences of { }, which do not occur inside formulas, and which are linearly ordered. A context with n holes is denoted by

$$\Gamma \underbrace{\{\ \} \ldots \{\ \}}_{n-\text{times}} \quad .$$

Holes can be filled with sequents, or contexts, in general. For example, if $\Gamma\{\ \}\{\ \} = A, [[B], \{\ \}], \{\ \}$ and $\Delta\{\ \} = C, [\{\ \}]$ then

$$\Gamma\{\Delta\{\ \}\}\{\ \} = A, [[B], C, [\{\ \}]], \{\ \} \quad ,$$

where in all contexts the holes are ordered from left to right as shown.

System $\mathsf{K} + \dot{\mathsf{X}}$**.** Figure 1 shows the set of rules from which we form our deductive systems. *System* K is the set of rules $\{\wedge, \vee, \Box, \mathsf{k}, \mathsf{ctr}\}$. We will look at extensions of System K with sets of rules $\dot{\mathsf{X}} \subseteq \{\dot{\mathsf{d}}, \dot{\mathsf{t}}, \dot{\mathsf{b}}, \dot{\mathsf{4}}, \dot{\mathsf{5}}\}$. The rules in $\dot{\mathsf{X}}$ are called *structural modal rules*. The $\dot{\mathsf{5}}$-rule is a bit special since it uses a two-hole-context. It can actually be decomposed into three rules that use unary contexts. However, here we prefer the presentation with a single rule. The $\dot{\mathsf{5}}$-rule may be understood as allowing to do the following, when going from premise to conclusion: take a boxed sequent $[\Delta]$, which is not at the root of the sequent, and move it to any other place in the sequent.

Notice that we have an explicit contraction rule in system K and that the k-rule is not invertible. It is of course easy to drop contraction and build it into the k-rule and into the rules in $\dot{\mathsf{X}}$, which makes all rules invertible. The reason we choose not to do this is because our cut-elimination procedure works better with explicit contraction.

There are also some rules that will turn out to be admissible, namely the $\Diamond$-rules, and the rules necessitation, weakening and cut, which are shown in Figure 2. A $\Diamond$-rule is in a certain sense the result of "reflecting" the corresponding structural rule at the cut, and vice versa. This comment will hopefully become more clear after the proof of the reduction lemma.

Given a set $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, $\dot{\mathsf{X}}$ is the corresponding subset of $\{\dot{\mathsf{d}}, \dot{\mathsf{t}}, \dot{\mathsf{b}}, \dot{\mathsf{4}}, \dot{\mathsf{5}}\}$ and $\mathring{\mathsf{X}}$ is the corresponding subset of $\{\mathring{\mathsf{d}}, \mathring{\mathsf{t}}, \mathring{\mathsf{b}}, \mathring{\mathsf{4}}, \mathring{\mathsf{5}}\}$.

Inference rules, derivations, proofs. In the following instance of an inference rule ρ

$$\rho \frac{\Gamma_1 \quad \ldots \quad \Gamma_n}{\Delta}$$

$$\Gamma\{p,\bar{p}\} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \mathsf{k}\ \frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathsf{ctr}\ \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\dot{\mathsf{d}}\ \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \dot{\mathsf{t}}\ \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \qquad \dot{\mathsf{b}}\ \frac{\Gamma\{[\Delta,[\Sigma]]\}}{\Gamma\{[\Delta],\Sigma\}}$$

$$\dot{\mathsf{4}}\ \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}} \qquad \dot{\mathsf{5}}\ \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}} \quad depth(\Gamma\{\ \}\{\emptyset\}) > 0$$

Fig. 1. System $\mathsf{K}+\{\dot{\mathsf{d}},\dot{\mathsf{t}},\dot{\mathsf{b}},\dot{\mathsf{4}},\dot{\mathsf{5}}\}$

$$\mathring{\mathsf{d}}\ \frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \mathring{\mathsf{t}}\ \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \mathring{\mathsf{b}}\ \frac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta,\Diamond A]\}}$$

$$\mathring{\mathsf{4}}\ \frac{\Gamma\{[\Delta,\Diamond A]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \mathring{\mathsf{5}}\ \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}} \quad depth(\Gamma\{\ \}\{\emptyset\}) > 0$$

$$\mathsf{nec}\ \frac{\Gamma}{[\Gamma]} \qquad \mathsf{wk}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Fig. 2. Diamond rules, necessitation, weakening, cut

we call $\Gamma_1 \ldots \Gamma_n$ its *premises* and Δ its *conclusion*. We write ρ^n to denote n instances of ρ and ρ^* to denote an unspecified number of instances of ρ. A *system*, denoted by $\mathcal{S}$, is a set of inference rules. A *derivation* in a system $\mathcal{S}$ is a finite tree whose nodes are labelled with sequents and which is built according to the inference rules from $\mathcal{S}$. The sequent at the root is the *conclusion* and the sequents at the leaves are the *premises* of the derivation. Derivations are denoted by $\mathcal{D}$. A derivation $\mathcal{D}$ with conclusion Γ in system $\mathcal{S}$ is sometimes shown as

$$\underset{\Gamma}{\bigtriangledown}\mathcal{S} \qquad .$$
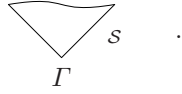
The depth of a derivation $\mathcal{D}$ is denoted by $|\mathcal{D}|$. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ and where all premises are instances of the axiom $\Gamma\{p,\bar{p}\}$. Proofs are denoted by $\mathcal{P}$. We write $\mathcal{S} \vdash \Gamma$ if there is a proof of Γ in system $\mathcal{S}$. An inference rule ρ is *(depth-preserving) admissible* for a system $\mathcal{S}$ if for each proof in $\mathcal{S} \cup \{\rho\}$ there is a proof of in $\mathcal{S}$ with the same conclusion (and with at most the same depth).

Soundness of our systems is easily established similarly to soundness of the systems in [4]:

Theorem 1 (Soundness). *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. *If a sequent is provable in* $\mathsf{K} + \dot{\mathsf{X}}$ *then its corresponding formula is provable in a Hilbert system for the modal logic* K *extended by the axioms in* X.

Our main result is cut-elimination, which we prove in the next section.

Theorem 2 (Cut-Elimination). *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. *If* $\mathsf{K} + \dot{\mathsf{X}} + \mathsf{cut} \vdash \Gamma$ *then* $\mathsf{K} + \dot{\mathsf{X}} \vdash \Gamma$.

By using cut-elimination we obtain the completeness theorem:

Theorem 3 (Completeness). *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. *If a formula is provable in a Hilbert system for the modal logic* K *extended by the modal axioms in* X *then it is provable in system* $\mathsf{K} + \dot{\mathsf{X}}$.

Proof. Given a proof in the Hilbert system we construct a proof in $\mathsf{K} + \dot{\mathsf{X}} + \mathsf{cut}$ as usual, and then apply Theorem 2 (Cut-elimination). We show proofs for the modal axioms:

$$
\begin{array}{l}
\mathsf{k}^2 \dfrac{[\bar{A}, A]}{\begin{array}{l}\dot{\mathsf{d}} \dfrac{\diamond\bar{A}, \diamond A, [\emptyset]}{\begin{array}{l}\vee \dfrac{\diamond\bar{A}, \diamond A}{\Box A \supset \diamond A}\end{array}}\end{array}}
\qquad
\mathsf{k} \dfrac{[A, \bar{A}]}{\begin{array}{l}\dot{\mathsf{t}} \dfrac{[\bar{A}], \diamond A}{\begin{array}{l}\vee \dfrac{\bar{A}, \diamond A}{A \supset \diamond A}\end{array}}\end{array}}
\qquad
\mathsf{k} \dfrac{[[A, \bar{A}]]}{\begin{array}{l}\dot{\mathsf{b}} \dfrac{[[\bar{A}], \diamond A]}{\begin{array}{l}\Box \dfrac{\bar{A}, [\diamond A]}{\begin{array}{l}\vee \dfrac{\bar{A}, \Box\diamond A}{A \supset \Box\diamond A}\end{array}}\end{array}}\end{array}}
\qquad
\mathsf{k} \dfrac{[\bar{A}, A], [\emptyset]}{\begin{array}{l}\dot{4} \dfrac{\diamond\bar{A}, [A], [\emptyset]}{\begin{array}{l}\Box^2 \dfrac{\diamond\bar{A}, [[A]]}{\begin{array}{l}\vee \dfrac{\diamond\bar{A}, \Box\Box A}{\Box A \supset \Box\Box A}\end{array}}\end{array}}\end{array}}
\qquad
\mathsf{k} \dfrac{[[\bar{A}, A]]}{\begin{array}{l}\dot{5} \dfrac{[[\bar{A}], \diamond A]}{\begin{array}{l}\Box^2 \dfrac{[\bar{A}], [\diamond A]}{\begin{array}{l}\vee \dfrac{\Box\bar{A}, \Box\diamond A}{\diamond A \supset \Box\diamond A}\end{array}}\end{array}}\end{array}} \quad .
\end{array}
$$

$\square$

3 Syntactic Cut-Elimination

We first need some definitions. The *depth* of a formula A, denoted $depth(A)$, is defined as usual, the depth of possibly negated atoms being zero. Given an instance of the cut rule as shown in Figure 2, its *cut formula* is A and its *cut rank* is one plus the depth of its cut formula. For $r \geq 0$ we define the rule cut_r which is cut with at most rank r. The *cut rank* of a derivation is the supremum of the cut ranks of its cuts. A rule is *cut-rank (and depth-) preserving admissible* for a system $\mathcal{S}$ if for all $r \geq 0$ the rule is (depth-preserving) admissible for $\mathcal{S} + \mathsf{cut}_r$.

Lemma 1 (Weakening and necessitation admissibility). *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. *The* wk*-rule and the* nec*-rule are depth- and cut-rank-preserving admissible for* $\mathsf{K} + \dot{\mathsf{X}}$.

Proof. A routine induction shows that a single nec or wk-rule can be eliminated from a given proof, a second induction on the number of nec or wk-rules yields our lemma. $\square$

$$\text{m}\square \; \frac{\Gamma\{[A,\ldots,A]\}}{\Gamma\{\square A\}} \qquad \text{m}\wedge \; \frac{\Gamma\{A,\ldots,A\} \quad \Gamma\{B,\ldots,B]\}}{\Gamma\{A \wedge B\}}$$

$$\text{mcut} \; \frac{\Gamma\{A,\ldots,A\} \quad \Gamma\{\bar{A},\ldots,\bar{A}\}}{\Gamma\{\emptyset\}} \qquad \text{med} \; \frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,\Sigma]\}} \qquad \text{fctr} \; \frac{\Gamma\{A,A\}}{\Gamma\{A\}}$$

Fig. 3. Multi-rules, medial, and formula contraction

Seriality (the rule $\dot{\text{d}}$) is different from the other rules: it trivially permutes below the cut. So we can get it out of the way and then prove cut elimination for the systems without seriality.

Lemma 2 (Push down seriality). *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$ *and* $\mathsf{d} \in \mathsf{X}$. *For each proof as shown on the left there is a proof as shown on the right:*
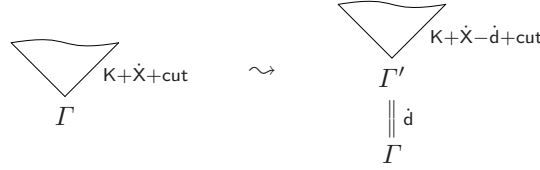
Proof. By an easy permutation argument, making use of weakening admissibility.

We also get contraction out of the way in order to eliminate the cut. First, we decompose contraction into the fctr-rule, which is contraction on formulas, and the med-rule, shown in Figure 3. We permute down the fctr-rule. It does not permute down below the rules cut, $\square$ and $\wedge$, so we generalise these rules as in Figure 3. We define a contraction-free system K^- as $\mathsf{K}^- = \mathsf{K} - \mathsf{ctr} + \{\mathsf{med}, \mathsf{m}\square, \mathsf{m}\wedge\}$ and will show cut elimination for that system, but first we develop the machinery to show that cut elimination for K^- leads to cut-elimination for K (with any $\dot{\mathsf{X}}$).

Lemma 3 (Decompose contraction). *The* ctr*-rule is derivable for* $\{\mathsf{fctr}, \mathsf{med}\}$.

Proof. By induction the depth of a sequent which is contracted, we show the inductive step:

$$\text{ctr} \; \frac{\Gamma\{A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n],A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]\}}{\Gamma\{A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]\}}$$

$$\rightsquigarrow \quad \text{med}^n \; \frac{\text{med}^n \; \frac{\Gamma\{A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n],A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]\}}{\text{ctr}^n \; \frac{\Gamma\{A_1,\ldots,A_m,A_1,\ldots,A_m,[\Delta_1,\Delta_1],\ldots,[\Delta_n,\Delta_n]\}}{\text{fctr}^m \; \frac{\Gamma\{A_1,\ldots,A_m,A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]\}}{\Gamma\{A_1,\ldots,A_m,[\Delta_1],\ldots,[\Delta_n]\}}}}}{}$$

□

Lemma 4 (Weakening and necessitation admissibility for K^-). *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. The wk-rule and the nec-rule are depth- and cut-rank-preserving admissible for $\mathsf{K}^- + \dot{\mathsf{X}}$.*

Lemma 5 (From mcut to cut). *The rule mcut_r is derivable for $\{\mathsf{cut}_r, \mathsf{wk}\}$.*

Proof. We define the rule $\mathsf{mcut}_r^{m,n}$ with $m, n > 0$ as

$$
\mathsf{mcut}_r^{m,n} \; \frac{\overbrace{\Gamma\{A, \dots, A\}}^{m-\text{times}} \quad \overbrace{\Gamma\{\bar{A}, \dots, \bar{A}\}}^{n-\text{times}}}{\Gamma\{\emptyset\}} \quad ,
$$

and show that rule derivable for $\{\mathsf{cut}_r, \mathsf{wk}\}$ by induction on $m + n$. The case for $m = n = 1$ is trivial, for $m > 1$ and $n = 1$ we replace

$$
\mathsf{mcut}_r^{m,1} \; \frac{\Gamma\{A, \dots, A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}
$$

by

$$
\mathsf{cut}_r \; \frac{\mathsf{mcut}_r^{m-1,1} \; \dfrac{\Gamma\{A, \dots, A\} \quad \mathsf{wk}\;\dfrac{\Gamma\{\bar{A}\}}{\Gamma\{\bar{A}, A\}}}{\Gamma\{A\}} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}
$$

and apply the induction hypothesis, and for $m, n > 1$ we replace

$$
\mathsf{mcut}_r^{m,n} \; \frac{\Gamma\{A, \dots, A\} \quad \Gamma\{\bar{A}, \dots, \bar{A}\}}{\Gamma\{\emptyset\}}
$$

by

$$
\mathsf{cut}_r \; \frac{\mathsf{mcut}_r^{m-1,n} \; \dfrac{\Gamma\{A, \dots, A\} \quad \mathsf{wk}\;\dfrac{\Gamma\{\bar{A}, \dots, \bar{A}\}}{\Gamma\{\bar{A}, \dots, \bar{A}, A\}}}{\Gamma\{A\}} \quad \mathsf{mcut}_r^{m,n-1} \; \dfrac{\mathsf{wk}\;\dfrac{\Gamma\{A, \dots, A\}}{\Gamma\{A, \dots, A, \bar{A}\}} \quad \Gamma\{\bar{A}, \dots, \bar{A}\}}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}
$$

and apply the induction hypothesis twice. $\qquad\square$

Lemma 6 (Push down contraction). *Let $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, 4, 5\}$. Given a proof as shown on the left, with ρ a single-premise-rule from $\mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{wk}$, there is a proof as shown on the right, with $|\mathcal{D}'| \leq |\mathcal{D}|$:*

$$
\rho \; \dfrac{\mathcal{D} \,\Big\|\, \mathsf{fctr} \; \dfrac{\overset{\mathcal{P}}{\diagdown}\; \mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{mcut} + \mathsf{wk}}{\Gamma_2}}{\dfrac{\Gamma_1}{\Gamma}} \qquad \leadsto \qquad \mathcal{D}' \,\Big\|\, \mathsf{fctr} \; \dfrac{\overset{\mathcal{P}'}{\diagdown}\; \mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{mcut} + \mathsf{wk}}{\Gamma_3}}{\Gamma} \quad .
$$

7

Proof. By induction on the length of $\mathcal{D}$ and a case analysis on ρ. Most cases are trivial. We show the two interesting ones. For $\rho = \vee$ and $\rho = \mathsf{k}$ we apply the following transformations:

$$\mathsf{fctr}\cfrac{\mathsf{fctr}\cfrac{\Gamma\{A, A, B\}}{\Gamma\{A, B\}}}{\vee\ \Gamma\{A \vee B\}} \qquad \rightsquigarrow \qquad \mathsf{fctr}\cfrac{\vee^2\cfrac{\mathsf{wk}\cfrac{\Gamma\{A, A, B\}}{\Gamma\{A, B, A, B\}}}{\Gamma\{A \vee B, A \vee B\}}}{\Gamma\{A \vee B\}}$$

$$\mathsf{k}\cfrac{\mathsf{fctr}\cfrac{\Gamma\{[A, A, \Delta]\}}{\Gamma\{[A, \Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad \rightsquigarrow \qquad \mathsf{fctr}\cfrac{\mathsf{k}^2\cfrac{\Gamma\{[A, A, \Delta]\}}{\Gamma\{\Diamond A, \Diamond A, [\Delta]\}}}{\Gamma\{\Diamond A, [\Delta]\}} \qquad ,$$

and in each case we apply the induction hypothesis twice. □

Proposition 1 (Push down contraction). *Given a proof as shown on the left, there is a proof as shown on the right:*

Proof. We first prove the claim that for each proof as shown on the left there is a proof as shown on the right:

The proof of the claim is by induction on the depth of $\mathcal{P}_1$, using Lemma 6 (Push down contraction). The proof of our proposition is as follows: by Lemma 3 (Decompose contraction) we obtain a proof in $\mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{cut} + \mathsf{fctr}$, we apply our claim, then we use Lemma 5 (From mcut to cut), to replace mcut, starting with the top-most instances. Finally we remove weakening using weakening admissibility. □

The following three lemmas are needed for the reduction lemma. We define

$$\mathsf{X}^+ = \begin{cases} \mathsf{X} \cup \{4\} & \text{if } \{\mathsf{t}, 5\} \subseteq \mathsf{X} \text{ or } \{\mathsf{b}, 5\} \subseteq \mathsf{X} \\ \mathsf{X} \cup \{5\} & \text{if } \{\mathsf{b}, 4\} \subseteq \mathsf{X} \\ \mathsf{X} & \text{otherwise} \end{cases} ,$$

and likewise for $\mathring{\mathsf{X}}$ and $\dot{\mathsf{X}}$.

Lemma 7 (Push down 45). *Let* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ *and* $\rho \in (\mathring{\mathsf{X}} \cap \{\mathring{\mathsf{4}}, \mathring{\mathsf{5}}\})$. *Given a derivation as shown on the left, where ρ applies to $\Diamond A$, there is a derivation as shown on the right, where all rules in $\mathcal{D}_3$ apply to the instance of $\Diamond A$ shown, and where $|\mathcal{D}_2| \leq |\mathcal{D}_1|$:*

$$
\rho\,\frac{\Gamma\{\Diamond A\}}{\Gamma_1\{\Diamond A\}} \\
\mathcal{D}_1 \,\Big\|\, \dot{\mathsf{X}}+\mathsf{med} \\
\Delta\{\Diamond A\}
\qquad \leadsto \qquad
\begin{array}{c}
\Gamma\{\Diamond A\} \\
\mathcal{D}_2 \,\Big\|\, \dot{\mathsf{X}}+\mathsf{med} \\
\Gamma_2\{\Diamond A\} \\
\mathcal{D}_3 \,\Big\|\, (\mathring{\mathsf{X}}^+ \cap \{\mathring{\mathsf{4}}, \mathring{\mathsf{5}}\}) \\
\Delta\{\Diamond A\}
\end{array}
\qquad .
$$

Proof. The proof is by induction on the length of $\mathcal{D}_1$. We permute the instance of ρ down and apply the induction hypothesis, possibly several times. We only show the non-trivial permutations.

$$
\mathsf{med}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\Gamma\{[\Diamond A, \Delta], [\Sigma]\}}{\Gamma\{\Diamond A, [\Delta], [\Sigma]\}}}{\Gamma\{\Diamond A, [\Delta, \Sigma]\}}
\qquad \leadsto \qquad
\mathring{\mathsf{4}}\,\frac{\mathsf{med}\,\dfrac{\Gamma\{[\Diamond A, \Delta], [\Sigma]\}}{\Gamma\{[\Diamond A, \Delta, \Sigma]\}}}{\Gamma\{\Diamond A, [\Delta, \Sigma]\}}
$$

$$
\dot{\mathsf{t}}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}}{\Gamma\{\Diamond A, \Delta\}}
\qquad \leadsto \qquad
\dot{\mathsf{t}}\,\frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, \Delta\}}
$$

$$
\dot{\mathsf{b}}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\Gamma\{[\Diamond A, \Delta, [\Sigma]]\}}}{\Gamma\{[\Diamond A, \Delta], \Sigma\}}
\qquad \leadsto \qquad
\mathring{\mathsf{5}}\,\frac{\dot{\mathsf{b}}\,\dfrac{\Gamma\{[\Delta, [\Diamond A, \Sigma]]\}}{\Gamma\{\Diamond A, [\Delta], \Sigma\}}}{\Gamma\{[\Diamond A, \Delta], \Sigma\}}
$$

$$
\dot{\mathsf{4}}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\Gamma\{[\Diamond A, \Delta], [\Sigma]\}}{\Gamma\{\Diamond A, [\Delta], [\Sigma]\}}}{\Gamma\{\Diamond A, [[\Delta], \Sigma]\}}
\qquad \leadsto \qquad
\mathring{\mathsf{4}}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\dot{\mathsf{4}}\,\dfrac{\Gamma\{[\Diamond A, \Delta], [\Sigma]\}}{\Gamma\{[[\Diamond A, \Delta], \Sigma]\}}}{\Gamma\{\Diamond A, [[\Delta], \Sigma]\}}}{\Gamma\{\Diamond A, [[\Delta], \Sigma]\}}
$$

$$
\dot{\mathsf{5}}\,\frac{\mathring{\mathsf{4}}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}\{\emptyset\}}{\Gamma\{\Diamond A, [\Delta]\}\{\emptyset\}}}{\Gamma\{\Diamond A\}\{[\Delta]\}}
\qquad \leadsto \qquad
\dot{\mathsf{5}}\,\frac{\mathring{\mathsf{5}}\,\dfrac{\Gamma\{[\Diamond A, \Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Diamond A, \Delta]\}}}{\Gamma\{\Diamond A\}\{[\Delta]\}}
$$

Permuting down the $\mathring{\mathsf{5}}$-rule is trivial except over the $\dot{\mathsf{t}}$-rule and the $\dot{\mathsf{b}}$-rule, and this is also trivial unless the restriction on the depth of the context in the $\mathring{\mathsf{5}}$-rule becomes relevant:

$$
\dot{\mathsf{t}}\,\frac{\mathring{\mathsf{5}}\,\dfrac{\Gamma_1, [\Delta], \Gamma_2\{\Diamond A\}}{\Gamma_1, [\Diamond A, \Delta], \Gamma_2\{\emptyset\}}}{\Gamma_1, \Diamond A, \Delta, \Gamma_2\{\emptyset\}}
\qquad \leadsto \qquad
\mathring{\mathsf{4}}^*\,\frac{\dot{\mathsf{t}}\,\dfrac{\Gamma_1, [\Delta], \Gamma_2\{\Diamond A\}}{\Gamma_1, \Delta, \Gamma_2\{\Diamond A\}}}{\Gamma_1, \Diamond A, \Delta, \Gamma_2\{\emptyset\}}
$$

9

$$
\mathring{5}\,\frac{[\Delta,[\Sigma]],\Gamma\{\Diamond A\}}{\dot{b}\,\dfrac{[\Delta,[\Sigma,\Diamond A]],\Gamma\{\emptyset\}}{[\Delta],\Sigma,\Diamond A,\Gamma\{\emptyset\}}}
\qquad\rightsquigarrow\qquad
\dot{b}\,\frac{[\Delta,[\Sigma]],\Gamma\{\Diamond A\}}{\mathring{4}*\,\dfrac{[\Delta],\Sigma,\Gamma\{\Diamond A\}}{[\Delta],\Sigma,\Diamond A,\Gamma\{\emptyset\}}}
$$

$\square$

Lemma 8 (Push down ktb**).** *Let* $\mathsf{X} \subseteq \{\mathsf{t},\mathsf{b},4,5\}$ *and let* $\rho = \mathsf{k}$ *or* $\rho \in (\mathring{\mathsf{X}} \cap \{\mathring{\mathsf{t}},\mathring{\mathsf{b}}\})$. *Given a derivation as shown on the left, where ρ applies to $\Diamond A$, there is a derivation as shown on the right, with $\sigma = \mathsf{k}$ or $\sigma \in (\mathring{\mathsf{X}} \cap \{\mathring{\mathsf{t}},\mathring{\mathsf{b}}\})$, where all rules in $\mathcal{D}_3$ apply to the instance of $\Diamond A$ shown, and where $|\mathcal{D}_2| \leq |\mathcal{D}_1|$:*

$$
\rho\,\frac{\Gamma\{A\}}{\Gamma_1\{\Diamond A\}}
\quad \mathcal{D}_1 \,\Big\|\, \dot{\mathsf{X}}+\mathsf{med}
\quad \frac{}{\Delta\{\Diamond A\}}
\qquad\rightsquigarrow\qquad
\begin{array}{c}
\Gamma\{A\} \\[2pt]
\mathcal{D}_2 \,\Big\|\, \dot{\mathsf{X}}+\mathsf{med} \\[2pt]
\sigma\,\dfrac{\Gamma_3\{A\}}{\Gamma_2\{\Diamond A\}} \\[2pt]
\mathcal{D}_3 \,\Big\|\, (\mathring{\mathsf{X}}^{+}\cap\{\mathring{4},\mathring{5}\}) \\[2pt]
\Delta\{\Diamond A\}
\end{array}
\;.
$$

Proof. The proof is by induction on the length of $\mathcal{D}_1$. We permute the instance of ρ down and apply Lemma 7 (Push down 45) and/or the induction hypothesis. We only show the non-trivial permutations.

$$
\mathsf{k}\,\frac{\Gamma\{[A,\Delta]\}}{\dot{\mathsf{t}}\,\dfrac{\Gamma\{\Diamond A,[\Delta]\}}{\Gamma\{\Diamond A,\Delta\}}}
\qquad\rightsquigarrow\qquad
\dot{\mathsf{t}}\,\frac{\Gamma\{[A,\Delta]\}}{\mathring{\mathsf{t}}\,\dfrac{\Gamma\{A,\Delta\}}{\Gamma\{\Diamond A,\Delta\}}}
$$

$$
\mathsf{k}\,\frac{\Gamma\{[\Delta,[A,\Sigma]]\}}{\dot{\mathsf{b}}\,\dfrac{\Gamma\{[\Diamond A,\Delta,[\Sigma]]\}}{\Gamma\{[\Diamond A,\Delta],\Sigma\}}}
\qquad\rightsquigarrow\qquad
\dot{\mathsf{b}}\,\frac{\Gamma\{[\Delta,[A,\Sigma]]\}}{\mathring{\mathsf{b}}\,\dfrac{\Gamma\{A,[\Delta],\Sigma\}}{\Gamma\{[\Diamond A,\Delta],\Sigma\}}}
$$

$$
\mathsf{k}\,\frac{\Gamma\{[A,\Delta],[\Sigma]\}}{\dot{4}\,\dfrac{\Gamma\{\Diamond A,[\Delta],[\Sigma]\}}{\Gamma\{\Diamond A,[[\Delta],\Sigma]\}}}
\qquad\rightsquigarrow\qquad
\dot{4}\,\frac{\Gamma\{[A,\Delta],[\Sigma]\}}{\mathsf{k}\,\dfrac{\Gamma\{[[A,\Delta],\Sigma]\}}{\mathring{4}\,\dfrac{\Gamma\{[\Diamond A,[\Delta],\Sigma]\}}{\Gamma\{\Diamond A,[[\Delta],\Sigma]\}}}}
$$

$$
\mathsf{k}\,\frac{\Gamma\{[A,\Delta]\}\{\emptyset\}}{\dot{5}\,\dfrac{\Gamma\{\Diamond A,[\Delta]\}\{\emptyset\}}{\Gamma\{\Diamond A\}\{[\Delta]\}}}
\qquad\rightsquigarrow\qquad
\dot{5}\,\frac{\Gamma\{[A,\Delta]\}\{\emptyset\}}{\mathsf{k}\,\dfrac{\Gamma\{\emptyset\}\{[A,\Delta]\}}{\mathring{5}\,\dfrac{\Gamma\{\emptyset\}\{\Diamond A,[\Delta]\}}{\Gamma\{\Diamond A\}\{[\Delta]\}}}}
$$

The cases for $\rho = \mathring{\mathsf{t}}$ are trivial.

10

$$\overset{\circ}{\mathsf{b}}\dfrac{\Gamma\{[\Delta],A\}}{\dot{\mathsf{t}}\dfrac{\Gamma\{[\Delta,\Diamond A]\}}{\Gamma\{\Delta,\Diamond A\}}}
\qquad\leadsto\qquad
\dot{\mathsf{t}}\dfrac{\Gamma\{[\Delta,A]\}}{\overset{\circ}{\mathsf{t}}\dfrac{\Gamma\{\Delta,A\}}{\Gamma\{\Delta,\Diamond A\}}}$$

$$\overset{\circ}{\mathsf{b}}\dfrac{\Gamma\{[\Sigma,[\Delta],A]\}}{\dot{\mathsf{b}}\dfrac{\Gamma\{[\Sigma,[\Delta,\Diamond A]]\}}{\Gamma\{[\Sigma],\Delta,\Diamond A\}}}
\qquad\leadsto\qquad
\dot{\mathsf{b}}\dfrac{\Gamma\{[\Sigma,[\Delta],A]\}}{\mathsf{k}\dfrac{\Gamma\{[\Sigma,A],\Delta\}}{\Gamma\{[\Sigma],\Delta,\Diamond A\}}}$$

$$\overset{\circ}{\mathsf{b}}\dfrac{\Gamma\{[\Delta],A,[\Sigma]\}}{\dot{4}\dfrac{\Gamma\{[\Delta,\Diamond A],[\Sigma]\}}{\Gamma\{[[\Delta,\Diamond A],\Sigma]\}}}
\qquad\leadsto\qquad
\dot{4}\dfrac{\Gamma\{[\Delta],A,[\Sigma]\}}{\overset{\circ}{\mathsf{b}}\dfrac{\Gamma\{[[\Delta],\Sigma],A\}}{\overset{\circ}{\mathsf{5}}\dfrac{\Gamma\{[[\Delta],\Diamond A,\Sigma]\}}{\Gamma\{[[\Delta,\Diamond A],\Sigma]\}}}}$$

For permuting down over the $\dot{\mathsf{5}}$-rule, in the only non-trivial case, notice that the context has to be of the form shown because of the restriction of context depth in the $\dot{\mathsf{5}}$-rule:

$$\overset{\circ}{\mathsf{b}}\dfrac{\Gamma\{\emptyset\}\{[\Sigma,[\Delta],A]\}}{\dot{\mathsf{5}}\dfrac{\Gamma\{\emptyset\}\{[\Sigma,[\Delta,\Diamond A]]\}}{\Gamma\{[\Delta,\Diamond A]\}\{[\Sigma,\emptyset]\}}}
\qquad\leadsto\qquad
\dot{\mathsf{5}}\dfrac{\Gamma\{\emptyset\}\{[\Sigma,[\Delta],A]\}}{\mathsf{k}\dfrac{\Gamma\{[\Delta]\}\{[A,\Sigma]\}}{\overset{\circ}{\mathsf{5}}\dfrac{\Gamma\{[\Delta]\}\{\Diamond A,[\Sigma,\emptyset]\}}{\Gamma\{[\Delta,\Diamond A]\}\{[\Sigma,\emptyset]\}}}}$$

□

Lemma 9 (Reflect 45). *Let* $\mathsf{X}\subseteq\{4,5\}$*. Given a derivation as shown on the left, where all rules in* $\mathcal{D}$ *apply to the instance of* $\Diamond A$ *shown, then for each sequent* Δ *there is a derivation as shown on the right:*

$$\begin{array}{c}\Gamma\{\Diamond A\}\{\emptyset\}\\[2pt]\mathcal{D}\;\Big\|\;\overset{\circ}{\mathsf{X}}\\[2pt]\Gamma\{\emptyset\}\{\Diamond A\}\end{array}
\qquad\leadsto\qquad
\begin{array}{c}\Gamma\{\emptyset\}\{[\Delta]\}\\[2pt]\mathcal{D}'\;\Big\|\;\dot{\mathsf{X}}\\[2pt]\Gamma\{[\Delta]\}\{\emptyset\}\end{array}\qquad.$$

Proof. By induction on the length of $\mathcal{D}$. □

Lemma 10 (Reduction Lemma). *Let* $\mathsf{X}\subseteq\{\mathsf{t},\mathsf{b},4,5\}$*. Given a proof as shown on the left, with* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *in* $\mathsf{K}^-+\dot{\mathsf{X}}+\mathsf{cut}_r$*, then there is a proof* $\mathcal{P}$ *in* $\mathsf{K}^-+\dot{\mathsf{X}}^++\mathsf{cut}_r$ *as shown on the right:*

$$\mathsf{cut}_{r+1}\dfrac{\begin{array}{cc}\overset{\displaystyle\mathcal{P}_1}{\Gamma_1\{A\}}&\overset{\displaystyle\mathcal{P}_2}{\Gamma_2\{\bar{A}\}}\\[2pt]\Big\|\,\dot{\mathsf{X}}+\mathsf{med}&\Big\|\,\dot{\mathsf{X}}+\mathsf{med}\\[2pt]\Gamma\{A\}&\Gamma\{\bar{A}\}\end{array}}{\Gamma\{\emptyset\}}
\qquad\leadsto\qquad
\begin{array}{c}\overset{\displaystyle\mathcal{P}}{}\\[2pt]\Gamma\{\emptyset\}\end{array}\qquad.$$

11

Proof. As usual, by an induction on $|\mathcal{P}_1| + |\mathcal{P}_2|$ and a case analysis on the lowermost rules in $\mathcal{P}_1$ and $\mathcal{P}_2$. We only show the most complicated case, in which we cut a box introduced by the m□-rule against a diamond introduced by k-rule. All other cases are much simpler. We have

$$
\mathsf{cut}_{r+1}\cfrac{
\mathsf{m\square}\cfrac{\Gamma_1\{[B,\ldots,B]\}}{\Gamma_1\{\square B\}}\;\Big\Vert\,\dot{\mathsf{X}}\mathsf{+med} \quad\atop \Gamma\{\square B\}
\qquad
\mathsf{k}\cfrac{\Gamma_2'\{[\bar{B},\Delta]\}}{\Gamma_2'\{\Diamond\bar{B},[\Delta]\}}\;\Big\Vert\,\dot{\mathsf{X}}\mathsf{+med} \quad\atop \Gamma\{\Diamond\bar{B}\}
}{\Gamma\{\emptyset\}}
$$

In the left subderivation we permute down the instance of m□ and on the right subderivation we apply Lemma 8 (Push ktb down) in order to obtain the following derivation, where $\Gamma\{\ \} = \Gamma\{\ \}\{\emptyset\}$. Note that the second hole in the binary context marks the position to which the $\Diamond\bar{B}$ is moved:

$$
\mathsf{cut}_{r+1}\cfrac{
\mathsf{m\square}\cfrac{\Gamma\{[B,\ldots,B]\}\{\emptyset\}}{\Gamma\{\square B\}\{\emptyset\}}
\qquad
\sigma\cfrac{\Gamma_3\{\bar{B}\}}{\Gamma\{\emptyset\}\{\Diamond\bar{B}\}}\;\Big\Vert\,(\mathsf{X}^+\cap\{4,5\})^\Diamond \atop \Gamma\{\Diamond\bar{B}\}\{\emptyset\}}
}{\Gamma\{\emptyset\}\{\emptyset\}}
$$

By using Lemma 9 (Reflect 45) we obtain a derivation $\mathcal{D}$ and build:

$$
\mathsf{cut}_{r+1}\cfrac{
\mathsf{m\square}\cfrac{\Gamma\{\emptyset\}\{[B,\ldots,B]\}}{\Gamma\{\emptyset\}\{\square B\}}
\qquad
\sigma\cfrac{\Gamma_3\{\bar{B}\}}{\Gamma\{\emptyset\}\{\Diamond\bar{B}\}}
}{\Gamma\{\emptyset\}\{\emptyset\}}\quad.
$$

We now consider the three possible cases for $\sigma \in \{\mathsf{k}, \mathring{\mathsf{t}}, \mathring{\mathsf{b}}\}$ and apply one of the following transformations to the relevant part of the proof:

$$
\mathsf{cut}_{r+1}\cfrac{
\mathsf{m\square}\cfrac{\Sigma\{[B,\ldots,B],[\Delta]\}}{\Sigma\{\square B,[\Delta]\}}
\quad
\mathsf{k}\cfrac{\Sigma\{[\bar{B},\Delta]\}}{\Sigma\{\Diamond\bar{B},[\Delta]\}}
}{\Sigma\{[\Delta]\}}
\quad\rightsquigarrow\quad
\mathsf{mcut}_r\cfrac{
\mathsf{med}\cfrac{\Sigma\{[B,\ldots,B],[\Delta]\}}{\Sigma\{[B,\ldots,B,\Delta]\}}
\quad
\Sigma\{[\bar{B},\Delta]\}
}{\Sigma\{[\Delta]\}}
$$

$$
\mathsf{cut}_{r+1}\cfrac{
\mathsf{m\square}\cfrac{\Sigma\{[B,\ldots,B]\}}{\Sigma\{\square B\}}
\quad
\mathring{\mathsf{t}}\cfrac{\Sigma\{\bar{B}\}}{\Sigma\{\Diamond\bar{B}\}}
}{\Sigma\{\emptyset\}}
\quad\rightsquigarrow\quad
\mathsf{mcut}_r\cfrac{
\mathring{\mathsf{t}}\cfrac{\Sigma\{[B,\ldots,B]\}}{\Sigma\{B,\ldots,B\}}
\quad
\Sigma\{\bar{B}\}
}{\Sigma\{\emptyset\}}
$$

12

$$
\mathsf{cut}_{r+1}\ \frac{\mathsf{m}\square\ \dfrac{\Sigma\{[[B,\ldots,B],\Delta]\}}{\Sigma\{[\square B,\Delta]\}} \qquad \mathsf{\lozenge}\ \dfrac{\Sigma\{\bar{B},[\Delta]\}}{\Sigma\{[\lozenge\bar{B},\Delta]\}}}{\Sigma\{[\Delta]\}} \quad \rightsquigarrow \quad \mathsf{mcut}_r\ \frac{\dot{\mathsf{b}}\ \dfrac{\Sigma\{[[B,\ldots,B],\Delta]\}}{\Sigma\{B,\ldots,B,\Delta]\}} \qquad \Sigma\{\bar{B},[\Delta]\}}{\Sigma\{[\Delta]\}} \qquad .
$$

We then eliminate mcut by using Lemma 5 (From mcut to cut) and weakening admissibility. $\qquad\square$

Proposition 2 (Cut-elimination for K^-). *Let* $\mathsf{X} \subseteq \{\mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *If* $\mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{cut} \vdash \Gamma$ *then* $\mathsf{K}^- + \dot{\mathsf{X}}^+ \vdash \Gamma$.

Proof. We first prove the claim: If $\mathsf{K}^- + \dot{\mathsf{X}} + \mathsf{cut}_{r+1} \vdash \Gamma$ then $\mathsf{K}^- + \dot{\mathsf{X}}^+ + \mathsf{cut}_r \vdash \Gamma$. The claim is proved by induction on the depth of the given proof, using the reduction lemma. Our proposition then follows from an induction on the cut rank of the given proof, using the claim. $\qquad\square$

Lemma 11 (From X^+ to X).
(i) The $\dot{\mathsf{4}}$-rule is derivable for $\{\dot{\mathsf{t}}, \dot{\mathsf{5}}, \mathsf{nec}\}$.
(ii) The $\dot{\mathsf{4}}$-rule is derivable for $\{\dot{\mathsf{b}}, \dot{\mathsf{5}}, \mathsf{nec}\}$.
(iii) The $\dot{\mathsf{5}}$-rule is derivable for $\{\dot{\mathsf{b}}, \dot{\mathsf{4}}, \mathsf{wk}\}$.

Proof. For (i) notice that the $\dot{\mathsf{4}}$-rule is a special case of the $\dot{\mathsf{5}}$-rule unless $\Gamma\{\ \}$ has depth zero, and thus $\Gamma\{\ \} = \Lambda, \{\ \}$. In that case we have:

$$
\dot{\mathsf{4}}\ \frac{\Lambda, [\Delta], [\Sigma]}{\Lambda, [[\Delta], \Sigma]} \qquad \rightsquigarrow \qquad \dot{\mathsf{t}}\ \frac{\dot{\mathsf{5}}\ \dfrac{\mathsf{nec}\ \dfrac{\Lambda, [\Delta], [\Sigma]}{[\Lambda, [\Delta], [\Sigma]]}}{[\Lambda, [[\Delta], \Sigma]]}}{\Lambda, [[\Delta], \Sigma]} \qquad .
$$

For (ii) we again have to consider only the case where $\Gamma\{\ \} = \Lambda, \{\ \}$:

$$
\dot{\mathsf{4}}\ \frac{\Lambda, [\Delta], [\Sigma]}{\Lambda, [[\Delta], \Sigma]} \qquad \rightsquigarrow \qquad \dot{\mathsf{b}}\ \frac{\dot{\mathsf{b}}\ \dfrac{\dot{\mathsf{5}}\ \dfrac{\mathsf{nec}^2\ \dfrac{\Lambda, [\Delta], [\Sigma]}{[[\Lambda, [\Delta], [\Sigma]]]}}{[[\Lambda, [\Sigma]], [\Delta]]}}{[[\Lambda], [\Delta], \Sigma]}}{\Lambda, [[\Delta], \Sigma]}
$$

For (iii) notice that a sequent has a tree structure and that, seen upwards, the $\dot{\mathsf{5}}$-rule allows to move a boxed sequent $[\Delta]$ to any position in that tree, but not to the root. To move a boxed sequent to any position in the tree it is enough if we are both able to move it a) from a given node the parent of this node and b) to move it from a given node to any child of that node. Point a) is just the

$\dot{4}$-rule and point b) is as follows:

$$\dot{\mathsf{b}} \cfrac{\dot{4} \cfrac{\mathsf{wk} \cfrac{\Gamma\{[\Lambda, [\Delta]]\}}{\Gamma\{[\Lambda, [\emptyset], [\Delta]]\}}}{\Gamma\{[\Lambda, [[\Delta]]]\}}}{\Gamma\{[\Lambda], [\Delta]\}} \qquad .$$

$\square$

Proof (of Theorem 2 (Cut-elimination)). We first prove the theorem for the cases where $\mathsf{d} \notin \mathsf{X}$. The transformation (i) is by Proposition 1 (Push down contraction), the transformation (ii) is Proposition 2 (Cut-elimination for K^-), and transformation (iii) is by Lemma 11 (From X^+ to X) and weakening and necessitation admissibility.
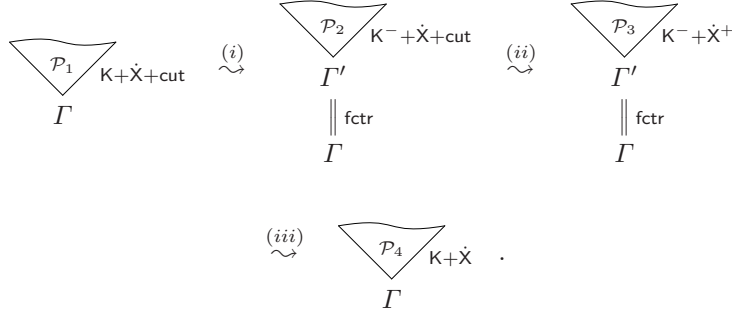
In the cases where $\mathsf{d} \in \mathsf{X}$ we first apply Lemma 2 (Push down seriality) and then proceed the same way with the upper part of the proof. $\square$

Future work. It looks like the cut-elimination proof can be generalised. So it is our goal to devise 1) easily checkable critera on rules, which guarantee cut-elimination, and 2) a procedure which turns modal axioms into rules which satisfy these criteria. Such a generic cut-elimination procedure exists already for the display calculus, but relies on the so-called display property, and thus on the language of tense logic. Recently, such a procedure has also been proposed by Ciabattoni et al. for hypersequent systems [8]. While hypersequents do not seem to be expressive enough for the modal logics considered here, nested sequents seem to add just the right amount of extra generality to enable a similar development for modal logics.

Related work. The fact that structural rules improve modularity has been observed before by Castilho et. al. [7] in the context of tableau systems. Our $\diamond$-rules correspond exactly to what are called propagation rules in [7]. While the focus of [7] is on giving decision procedures, our focus is on giving proof systems which support a notion of cut-elimination. This is more easily done with local rules, so in sequent systems instead of tableau systems. A further difference is that propagation rules and structural rules are mixed in [7], while here we treat systems with structural rules only (and in [3] we treated systems with propagation- or $\diamond$-rules only).

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