Syntactic Cut-Elimination for Common Knowledge

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Abstract

We first look at an existing infinitary sequent system for common knowledge for which there is no known syntactic cut-elimination procedure and also no known non-trivial bound on the proof-depth. We then present another infinitary sequent system based on *nested sequents* that are essentially trees and with inference rules that apply deeply inside of these trees. Thus we call this system "deep" while we call the former system "shallow". In contrast to the shallow system, the deep system allows to give a straightforward syntactic cut-elimination procedure. Since both systems can be embedded into each other, this also yields a syntactic cut-elimination procedure for the shallow system. For both systems we thus obtain an upper bound of $\varphi_2 0$ on the depth of proofs, where φ is the Veblen function.

Key words: cut elimination, infinitary sequent system, nested sequents, common knowledge

1. Introduction

The notion of *common knowledge* is well-studied in epistemic logic, where modalities express knowledge of agents. Two standard textbooks on epistemic logic and common knowledge in particular, are [7] by Fagin, Halpern, Moses, and Vardi and [13] by Meyer and van der Hoek.

The fact that a proposition A is common knowledge can be expressed by the infinite conjunction "all agents know A and all agents know that all agents know A and so on". In order to express this in a finite way we can use fixpoints: common knowledge of A is then defined to be the greatest fixpoint of the function

$$X \mapsto \text{ everybody knows } A \text{ and everybody knows } X.$$

Such a definition was introduced by Halpern and Moses [9] and further studied in [7].

The traditional way to formalise common knowledge is to use a Hilbert-style axiom system. Such a system has a fixpoint axiom, which states that common knowledge is a fixpoint, and an induction rule, which states that this fixpoint is the greatest fixpoint. However, this approach does not work well for designing a Gentzen-style sequent calculus. In particular, Alberucci and Jäger show in [2] that a cut-free sequent system designed in this way is not complete.

To obtain a complete cut-free system Alberucci and Jäger replace the induction rule by an infinitary ω-rule. This results in a system in which proofs have transfinite depth and in which common knowledge is the greatest fixpoint of the function described above. Although this system has been further studied in [12, 10], no syntactic cut-elimination procedure has been found. Cut-elimination was proved only indirectly by showing completeness of the cut-free system. No non-trivial bound on the depth of proofs in this system is known.

In the present paper, we give a syntactic cut-elimination procedure for another infinitary system of common knowledge. It uses *nested sequents* which are essentially trees and its inference rules apply deeply inside of these trees. Thus we call this system "deep" while we call the system by Alberucci and Jäger "shallow". The deep system allows to straightforwardly apply the method of *predicative cut-elimination*, which is a standard tool for the proof-theoretic analysis of systems of set theory and second order number theory, see Pohlers [15, 16] and Schütte [18]. Since the shallow and the deep system can be embedded into each other, this also yields a syntactic cut-elimination procedure for the shallow system. For both systems we thus obtain an upper bound of $\varphi_2 0$ on the depth of proofs, where φ is the Veblen function.

Please note that, like Alberucci and Jäger, our term *logic of common knowledge* refers to the least normal modal logic K, with an added fixpoint modality. Some people might prefer to call that the logic of common *belief*. We think that the methods introduced here transfer easily to cases where modal axioms like reflexivity, transitivity, and even symmetry, are added. Cut elimination results for these modal logics without common knowledge can be found in [6]. The combination of the techniques presented here and the ones in [6] should suffice to get cut elimination for modal logics with additional modal axioms and common knowledge.

The general idea of applying rules deeply has been proposed several times in different forms and for different purposes. Schütte already used it in order to obtain systems without contraction and weakening, which he considered more elegant [17]. Guglielmi used it to give a proof-theoretic system for a certain substructural logic which cannot be captured in the sequent calculus. To do so, he developed the *calculus of structures*, a formalism which is centered around deep inference and abolishes the traditional format of sequent calculus proofs [8]. The calculus of structures then has also been developed for modal logic [19]. Based on these ideas, Brünnler introduced the notion of deep sequent (now called nested sequent) and gave a systematic set of sequent systems and a corresponding cut-elimination procedure for the modal logics between K and S5 [6]. Kashima had used the same notion of nested sequent already in [11] in order to give cut-free sequent systems for some tense logics.

Several cut-free systems for logics with common knowledge exist already. The one that is closest to our system was introduced by Tanaka in [20] for predicate common knowledge logic and is based on Kashima's ideas. It essentially also uses what we call deep sequents. In fact, if one disregards the rather different notation and some choices in the formulation of rules, then one could say that our system is the propositional part of Tanaka's system. There are also finitary systems. Abate, Goré and Widmann, for example, introduce a cut-free tableau system for common knowledge in [1]. Cut-free system have also been studied in the context of *explicit modal logic* by Artemov [4] and by Antonakos [3].

However, we do not know of syntactic cut-elimination procedures for any of the systems mentioned. Typically, cut-elimination is established only indirectly. There are cut-elimination procedures for similar logics, for example by Pliuškevičius for an infinitary system for linear time temporal logic in [14]. For linear temporal logic there is no need for nested sequents. For this logic it is enough to use indexed formulas of the form A^i which denotes A at the i-th moment in time.

The paper is organised as follows. We first review the shallow sequent system by Alberucci and Jäger and show the obstacle to cut-elimination. We then present our deep sequent system, prove the invertibility of its rules, the admissibility of the structural rules and finally cut-elimination. Then we embed the shallow system into the deep system and vice versa, thus establishing cut-elimination for the shallow system. Then, by embedding the Hilbert system into our deep sequent system, we obtain an upper bound for the depth of proofs in both the shallow and the deep system. Some discussion about future work ends this paper.

This is the journal version of [5]. Some definitions and proofs are included which were omitted in [5]. The main difference, however, is the inclusion of the relationship with the shallow sequent system (Section 2 and 4).

2. The Shallow Sequent System

Formulas and sequents. We are considering a language with h agents for some $h > 0$. Propositions p and their negations $\bar{p}$ are *atoms*, with $\bar{\bar{p}}$ defined to be p. *Formulas* are denoted by A, B, C, D. They are given by the following grammar:

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond_i A \mid \Box_i A \mid \circledast\!\!\Diamond A \mid \boxast A \quad ,$$

where $1 \leq i \leq h$. The formula $\Box_i A$ is read as "agent i knows A" and the formula $\boxast A$ is read as "A is common knowledge". The connectives $\Box_i$ and $\boxast$ have $\Diamond_i$ and $\circledast\!\!\Diamond$ as their respective De Morgan duals. Binary connectives are left-associative: $A \vee B \vee C$ denotes $((A \vee B) \vee C)$, for example.

Given a formula A, its *negation* $\bar{A}$ is defined as usual using the De Morgan laws, $A \supset B$ is defined as $\bar{A} \vee B$ and $\bot$ is defined as $p \wedge \bar{p}$ for some proposition p. The formula $\Box A$ is an abbreviation for "everybody knows A":

$$\Box A = \Box_1 A \wedge \ldots \wedge \Box_h A \qquad \text{and} \qquad \Diamond A = \Diamond_1 A \vee \ldots \vee \Diamond_h A.$$

A sequence of $n \geq 0$ modal connectives can be abbreviated, for example

$$\Box^n A = \underbrace{\Box \ldots \Box}_{n-\text{times}} A \quad .$$

A *(shallow) sequent* is a finite multiset of formulas. Sequents are denoted by $\Gamma, \Delta, \Lambda, \Pi, \Sigma$.

Inference rules. In an instance of the inference rule ρ

$$\rho \, \frac{\Gamma_1 \quad \Gamma_2 \quad \ldots}{\Delta}$$

the sequents $\Gamma_1, \Gamma_2 \dots$ are its *premises* and the sequent Δ is its *conclusion*. An *axiom* is a rule without premises. We will not distinguish between an axiom and its conclusion. A *system*, denoted by $\mathcal{S}$, is a set of rules. Figure 1 shows system $\mathsf{G_C}$, a shallow sequent calculus for the logic of common knowledge. Its only axiom is called *identity axiom*. Notice that the $\boxast$-rule has infinitely many premises. If Γ is a sequent then $\Diamond_i \Gamma$ is obtained from Γ by prefixing the connective $\Diamond_i$ to each formula occurrence in Γ, and similarly for other connectives.

$$\Gamma, p, \bar{p} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box_i\,\frac{\Gamma, \diamondast\Delta, A}{\Diamond_i \Gamma, \diamondast\Delta, \Box_i A, \Sigma}$$

$$\boxast\,\frac{\Gamma, \Box^k A \quad \text{for all } k \geq 1}{\Gamma, \boxast A} \qquad \diamondast\,\frac{\Gamma, \diamondast A, \Diamond A}{\Gamma, \diamondast A}$$

Figure 1: System $\mathsf{G_C}$

$$\mathsf{wk}\,\frac{\Gamma}{\Gamma, A} \qquad \mathsf{ctr}\,\frac{\Gamma, A, A}{\Gamma, A} \qquad \mathsf{cut}\,\frac{\Gamma, A \quad \Delta, \bar{A}}{\Gamma, \Delta}$$

Figure 2: Weakening, contraction and cut for system $\mathsf{G_C}$

Derivations and proofs. In the following, a *tree* is a tree in the graph-theoretic sense, and may be infinite. A tree is *well-founded* if it does not have an infinite path. A *derivation* in a system $\mathcal{S}$ is a directed, rooted, ordered and well-founded tree whose nodes are labelled with sequents and which is built according to the inference rules from $\mathcal{S}$. Derivations are visualised as upward-growing trees, so the root is at the bottom. The sequent at the root is the *conclusion* and the sequents at the leaves are the *premises* of the derivation. A *proof* of a sequent Γ in a system is a derivation in this system with conclusion Γ where all leaves are axioms. Proofs are denoted by π. We write $\mathcal{S} \vdash \Gamma$ if there is a proof of Γ in system $\mathcal{S}$. Given a proof π we denote its depth by $|\pi|$. Notice that derivations here are in general infinitely branching, thus their depth can be infinite even though each branch has to be finite.

Formula rank. Notice that formulas in the premises of the $\boxast$-rule are generally larger than formulas in its conclusion. This is typically a problem for cut-elimination, but we can easily solve this by defining an appropriate measure. For a formula A we define its *rank* $rk(A)$ as follows:

$$
\begin{aligned}
&rk(p) = rk(\bar{p}) = 0\\
&rk(A \wedge B) = rk(A \vee B) = max(rk(A), rk(B)) + 1\\
&rk(\Box_i A) = rk(\Diamond_i A) = rk(A) + 1\\
&rk(\boxast A) = rk(\diamondast A) = \omega + rk(A)
\end{aligned}
$$

Lemma 1 (Some properties of the rank)**.** *For all formulas A we have that*
(i) $rk(A) = rk(\bar{A})$,
(ii) $rk(A) < \omega^2$,
(iii) for all $k < \omega$ we have $rk(\Box^k A) < rk(\boxast A)$.

Proof. Statements (i) and (ii) are immediate. For (iii), an induction on k yields that $rk(\Box^k A) = rk(A) + k \cdot h$. By (ii) it is then enough to check that for all k and all $\alpha < \omega^2$ we have $\alpha + k \cdot h < \omega + \alpha$. $\Box$

Cut rank. The *cut rank* of an instance of cut as shown in Figure 2 is the rank of its *cut formula A*. For an ordinal γ we define the rule cut_γ which is cut with at most rank γ and the rule $\mathsf{cut}_{<\gamma}$ which is cut with a rank strictly smaller than γ. For a system $\mathcal{S}$ and ordinals α and γ and a sequent Γ we write $\mathcal{S} \vdash^{\alpha}_{\gamma} \Gamma$ to say that there is a proof of Γ in system $\mathcal{S} + \mathsf{cut}_{<\gamma}$ with depth bounded by α. We write $\mathcal{S} \vdash^{<\alpha}_{\gamma} \Gamma$ to say that there is an ordinal $\alpha_0 < \alpha$ such that $\mathcal{S} \vdash^{\alpha_0}_{\gamma} \Gamma$.

Admissibility and invertibility. An inference rule ρ is *depth- and cut-rank-preserving admissible* or, for short, *perfectly admissible* for a system $\mathcal{S}$ if for each instance of ρ with premises $\Gamma_1, \Gamma_2 \ldots$ and conclusion Δ, whenever $\mathcal{S} \vdash^{\alpha}_{\gamma} \Gamma_i$ for each premise Γ_i then $\mathcal{S} \vdash^{\alpha}_{\gamma} \Delta$. For each rule ρ there is its *inverse*, denoted by $\bar{\rho}$, which has the conclusion of ρ as its only premise and any premise of ρ as its conclusion. An inference rule ρ is *perfectly invertible* for a system $\mathcal{S}$ if $\bar{\rho}$ is perfectly admissible for $\mathcal{S}$.

We omit the proof of the following lemma, which is standard.

Lemma 2 (Admissibility of the structural rules and invertibility)**.**
(i) The rules weakening and contraction from Figure 2 are perfectly admissible for system $\mathsf{G_C}$.

(ii) All rules of $\mathsf{G_C}$ *except for the* $\Box_i$*-rule are perfectly invertible for system* $\mathsf{G_C}$.

2.1. The Problem for Cut-Elimination

Let us look at the problem of cut-elimination in system $\mathsf{G_C}$. Consider the following proof:

$$
\mathsf{cut}\ \dfrac{\Box_i\ \dfrac{\begin{array}{c}\pi_1\\ A,\Gamma,\diamondast\bar{B}\end{array}}{\Box_i A,\Diamond_i\Gamma,\Sigma,\diamondast\bar{B}} \qquad \boxast\ \dfrac{\vdots \qquad \begin{array}{c}\pi_{2k}\\ \Box^k B,\Delta\end{array} \qquad \vdots}{\boxast B,\Delta}\ {\scriptstyle 1\le k<\omega}}{\Box_i A,\Diamond_i\Gamma,\Sigma,\Delta}
$$

Here the inference rule above the cut on the left does not apply to the cut formula while the inference rule on the right does. The typical transformation would push the left rule instance below the cut, as follows:

$$
\Box_i\ \dfrac{\mathsf{cut}\ \dfrac{\begin{array}{c}\pi_1\\ A,\Gamma,\diamondast\bar{B}\end{array} \qquad \boxast\ \dfrac{\vdots \qquad \begin{array}{c}\pi_{2k}\\ \Box^k B,\Delta\end{array} \qquad \vdots}{\boxast B,\Delta}\ {\scriptstyle 1\le k<\omega}}{A,\Gamma,\Delta}}{\Box_i A,\Diamond_i\Gamma,\Sigma,\Diamond_i\Delta}
$$

However, this transformation introduces the $\Diamond_i$ in $\Diamond_i\Delta$, and thus it does not yield a proof of the original conclusion. This is caused by the context restriction in the $\Box_i$-rule.

Such a context restriction also occurs in the standard sequent calculus for the modal logic K. While it destroys invertibility, at least it does not cause any difficulties for syntactic cut-elimination for K. However, we see that the context restriction poses a genuine problem for logics with more modalities like in the logic of common knowledge. In the next section we will see how a more general format for sequents and inference rules solves the problem since it does not require context restrictions.

3. The Deep Sequent System

Nested sequents. A *nested sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]_i$ where Γ is a nested sequent and $1 \leq i \leq h$. The letters $\Gamma, \Delta, \Lambda, \Pi, \Sigma$ from now on denote nested sequents and the word sequent from now on refers to nested sequent, except when it is clear from the context that a sequent is shallow, such as a sequent appearing in a derivation in $\mathsf{G_C}$. A sequent is always of the form

$$A_1, \dots, A_m, [\Delta_1]_{i_1}, \dots, [\Delta_n]_{i_n} \quad ,$$

where the i_j denote agents and thus range from 1 to h. As usual, the comma denotes multiset union and there is no distinction between a singleton multiset and its element.

Fix an arbitrary linear order on formulas. Fix an arbitrary linear order on boxed sequents. The *corresponding formula* of a non-empty sequent Γ, denoted $\underline{\Gamma}_{\mathsf{F}}$, is defined as follows:

$$\underline{A_1, \dots, A_m, [\Delta_1]_{i_1}, \dots, [\Delta_n]_{i_n}}_{\mathsf{F}} = A_1 \vee \cdots \vee A_m \vee \Box_{i_1} \underline{\Delta_1}_{\mathsf{F}} \vee \cdots \vee \Box_{i_n} \underline{\Delta_n}_{\mathsf{F}} \ ,$$

where formulas and boxed sequents are listed according to the fixed orders. The *corresponding formula* of the empty sequent is $\bot$. A sequent has a *corresponding tree* whose nodes are marked with multisets of formulas and whose edges are marked with agents. The corresponding tree of the above sequent is

$$\begin{tikzcd}
 & & \{A_1, \dots, A_m\} \arrow[dll, "i_1"'] \arrow[dl, "i_2"] \arrow[dr, "i_{n-1}"'] \arrow[drr, "i_n"] & & \\
\mathit{tree}(\Delta_1) & \mathit{tree}(\Delta_2) & \cdots & \mathit{tree}(\Delta_{n-1}) & \mathit{tree}(\Delta_n)
\end{tikzcd} \quad ,$$

where $\mathit{tree}(\Delta_1) \dots \mathit{tree}(\Delta_n)$ are the corresponding trees of $\Delta_1 \dots \Delta_n$. Often we do not distinguish between a sequent and its corresponding tree, e.g. the *root* of a sequent is the root of its corresponding tree.

Formula contexts and Sequent contexts. A *formula context* is a formula with exactly one occurrence of the special atom $\{\ \}$, which is called *the hole* or *the empty context*. A *sequent context* is a sequent with exactly one occurrence of the hole, which does not occur inside formulas. Formula contexts are denoted by $A\{\ \}, B\{\ \}$, and so on. Sequent contexts are denoted by $\Gamma\{\ \}$, $\Delta\{\ \}$, and so on. The formula $A\{B\}$ is obtained by replacing $\{\ \}$ inside $A\{\ \}$

$$
\Gamma\{p,\bar{p}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A \vee B\}}
$$

$$
\Box_i\,\frac{\Gamma\{[A]_i\}}{\Gamma\{\Box_i A\}} \qquad \Diamond_i\,\frac{\Gamma\{\Diamond_i A, [\Delta, A]_i\}}{\Gamma\{\Diamond_i A, [\Delta]_i\}}
$$

$$
\boxast\,\frac{\Gamma\{\Box^k A\} \quad \text{for all } k \geq 1}{\Gamma\{\boxast A\}} \qquad \diamondast\,\frac{\Gamma\{\diamondast A, \Diamond^k A\}}{\Gamma\{\diamondast A\}}
$$

Figure 3: System $\mathsf{D_C}$

$$
\mathsf{nec}\,\frac{\Gamma}{[\Gamma]_i} \qquad \mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{cut}\,\frac{\Gamma\{A\} \qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}
$$

Figure 4: Necessitation, weakening, contraction and cut for system $\mathsf{D_C}$

by B and the sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\ \}$ inside $\Gamma\{\ \}$ by Δ. For example, if $\Gamma\{\ \} = A, [[B]_1, \{\ \}]_2$ and $\Delta = C, [D]_3$ then

$$
\Gamma\{\Delta\} = A, [[B]_1, C, [D]_3]_2 \quad .
$$

The *corresponding formula context* of a sequent context $\Gamma\{\ \}$, denoted $\underline{\Gamma}_{\mathsf{F}}\{\ \}$ is defined as follows:

$$
\begin{aligned}
\underline{\Gamma, \{\ \}}_{\mathsf{F}} &= \underline{\Gamma}_{\mathsf{F}} \vee \{\ \} \\
\underline{\Gamma, [\Delta\{\ \}]_i}_{\mathsf{F}} &= \underline{\Gamma}_{\mathsf{F}} \vee \Box_i \underline{\Delta\{\ \}}_{\mathsf{F}}
\end{aligned}
$$

Figure 3 shows our deep sequent system $\mathsf{D_C}$. Figure 4 shows the structural rules *necessitation*, *weakening* and *contraction* as well as the rule *cut*, which are associated to system $\mathsf{D_C}$. Notice that the rules of system $\mathsf{D_C}$ and the associated rules are different from the corresponding rules in system $\mathsf{G_C}$ but have the same names. If we refer to a rule only by its name then it will be clear from the context which rule is meant. For example the cut in $\mathsf{G_C} + \mathsf{cut}$ is the one associated to system $\mathsf{G_C}$ and the one in $\mathsf{D_C} + \mathsf{cut}$ is the one associated with system $\mathsf{D_C}$.

Lemma 3 (Admissibility of the structural rules and invertibility)**.**
(i) The rules necessitation, weakening and contraction from Figure 4 are perfectly admissible for system $\mathsf{D_C}$.
(ii) All rules in $\mathsf{D_C}$ *are perfectly invertible for* $\mathsf{D_C}$.

Proof. Admissibility of necessitation and weakening follow from a routine induction on the depth of the proof. The same works for the invertibility of the $\wedge, \vee, \Box_i$ and $\boxast$-rules in (ii). The inverses of all other rules are just weakenings. For admissibility of contraction we also proceed by induction on the depth of the proof tree, using invertibility of the rules. The cases for the propositional rules and for the $\Box_i, \boxast, \diamondast$-rules are trivial. For the $\Diamond_i$-rule we consider the formula $\Diamond_i A$ from its conclusion $\Gamma\{\Diamond_i A, [\Delta]_i\}$ and its position inside the premise of contraction $\Lambda\{\Sigma, \Sigma\}$. We have the cases 1) $\Diamond_i A$ is inside Σ or 2) $\Diamond_i A$ is inside $\Lambda\{\ \}$. We have two subcases for case 1: 1.1) $[\Delta]_i$ inside $\Lambda\{\ \}$, 1.2) $[\Delta]_i$ inside Σ. There are three subcases of case 2: 2.1) $[\Delta]_i$ inside $\Lambda\{\ \}$ and 2.2) $[\Delta]_i$ inside Σ, 2.3) Σ, Σ inside $[\Delta]_i$. All cases are either simpler than or similar to case 2.2, which is as follows:

$$
\mathsf{ctr}\,\dfrac{\Diamond_i\,\dfrac{\Lambda'\{\Diamond_i A, \Sigma', [\Delta, A]_i, \Sigma', [\Delta]_i\}}{\Lambda'\{\Diamond_i A, \Sigma', [\Delta]_i, \Sigma', [\Delta]_i\}}}{\Lambda'\{\Diamond_i A, \Sigma', [\Delta]_i\}}
\quad\rightsquigarrow\quad
\Diamond_i\,\dfrac{\mathsf{ctr}\,\dfrac{\bar{\Diamond}_i\,\dfrac{\Lambda'\{\Diamond_i A, \Sigma', [\Delta, A]_i, \Sigma', [\Delta]_i\}}{\Lambda'\{\Diamond_i A, \Sigma', [\Delta, A]_i, \Sigma', [\Delta, A]_i\}}}{\Lambda'\{\Diamond_i A, \Sigma', [\Delta, A]_i\}}}{\Lambda'\{\Diamond_i A, \Sigma', [\Delta]_i\}}
\quad ,
$$

where the instance of $\bar{\Diamond}_i$ in the proof on the right is removed because it is perectly admissible and the instance of contraction is removed by the induction hypothesis. ∎

Lemma 4 (Admissibility of the general identity axiom)**.** *For all contexts* $\Gamma\{\ \}$ *and all formulas* A *we have* $\mathsf{D_C} \vdash^{2\cdot rk(A)}_{0} \Gamma\{A, \bar{A}\}$.

Proof. We perform an induction on $rk(A)$ and a case analysis on the main connective of A. The cases for atoms and for the propositional connectives are obvious. For $A = \Box_i B$ and $A = \boxast B$ we respectively have

$$\Box_i \dfrac{\Diamond_i \dfrac{\Gamma\{[B,\bar{B}]_i, \Diamond_i \bar{B}\}}{\Gamma\{[B]_i, \Diamond_i \bar{B}\}}}{\Gamma\{\Box_i B, \Diamond_i \bar{B}\}} \qquad \text{and} \qquad \boxast \dfrac{\vdots \quad \mathsf{wk}, \diamondast \dfrac{\Gamma\{\Box^k B, \Diamond^k \bar{B}\}}{\Gamma\{\Box^k B, \diamondast \bar{B}\}} \quad \vdots_{\;1 \leq k < \omega}}{\Gamma\{\boxast B, \diamondast \bar{B}\}} \quad .$$

On the left by induction hypothesis we get a proof of the premise of depth $2 \cdot rk(B)$ and thus a proof of the conclusion of depth $2 \cdot rk(B) + 2 = 2 \cdot (rk(B) + 1) = 2 \cdot rk(\Box_i B)$. On the right by Lemma 1 we can apply the induction hypothesis for each premise to get a proof of depth $2 \cdot rk(\Box^k B) = 2 \cdot (rk(B) + k \cdot h)$ and thus a proof of the conclusion of depth $2 \cdot (rk(B) + \omega) \leq 2 \cdot (\omega + rk(B)) = 2 \cdot rk(\boxast B)$. $\Box$

3.1. Cut-Elimination

We write $\alpha \# \beta$ for the *natural sum of* α *and* β which, in contrast to the ordinary ordinal sum, does not cancel additive components. For an introduction to ordinals, and a definition of the natural sum in particular, we refer to Schütte [18]. The *binary Veblen function* φ is generated inductively as follows:

1. $\varphi_0 \beta := \omega^\beta$,
2. if $\alpha > 0$, then $\varphi_\alpha \beta$ denotes the βth common fixpoint of the functions $\lambda \xi . \varphi_\gamma \xi$ for $\gamma < \alpha$.

In this subsection we write $\vdash^{\alpha}_{\beta} \Gamma$ for $\mathsf{D_C} \vdash^{\alpha}_{\beta} \Gamma$.

Lemma 5 (Reduction Lemma)**.** *If there is a proof*

$$\mathsf{cut}_\gamma \dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}$$

with π_1 *and* π_2 *in* $\mathsf{D_C} + \mathsf{cut}_{<\gamma}$ *then* $\vdash^{|\pi_1| \# |\pi_2|}_{\gamma} \Gamma\{\emptyset\}$ *.*

Proof. By induction on $|\pi_1| \# |\pi_2|$. We perform a case analysis on the two lowermost rules in the given proofs. If one of the two rules is passive and an axiom then $\Gamma\{\emptyset\}$ is axiomatic as well. If one is active and an axiom then we have

$$\mathsf{cut}_0\,\dfrac{\Gamma\{p,\bar{p}\} \qquad \overset{\pi_2}{\Gamma\{\bar{p},\bar{p}\}}}{\Gamma\{\bar{p}\}} \qquad \rightsquigarrow \qquad \mathsf{ctr}\,\dfrac{\overset{\pi_2}{\Gamma\{\bar{p},\bar{p}\}}}{\Gamma\{\bar{p}\}} \quad ,$$

and by contraction admissibility we have $\vdash^{|\pi_2|}_{0} \Gamma\{\bar{p}\}$ and thus $\vdash^{|\pi_1| \# |\pi_2|}_{0} \Gamma\{\bar{p}\}$. If some rule ρ is passive then we have

$$\mathsf{cut}_\gamma\,\dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad \rho\,\dfrac{\vdots \qquad \overset{\pi_{2i}}{\Gamma_i\{\bar{A}\}} \qquad \vdots}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}} \qquad \rightsquigarrow$$

$$\rho\,\dfrac{\vdots \qquad \mathsf{cut}_\gamma\,\dfrac{\bar{\rho}\,\dfrac{\overset{\pi_1}{\Gamma\{A\}}}{\Gamma_i\{A\}} \qquad \overset{\pi_{2i}}{\Gamma_i\{\bar{A}\}}}{\Gamma_i\{\emptyset\}} \qquad \vdots}{\Gamma\{\emptyset\}} \quad ,$$

where i ranges from 1 to the number of premises of ρ. By invertibility of ρ we get $\vdash^{|\pi_1|}_{\gamma} \Gamma_i\{A\}$, thus by induction hypothesis $\vdash^{|\pi_1| \# |\pi_{2i}|}_{\gamma} \Gamma_i\{\emptyset\}$ for all i and by ρ we get $\vdash^{|\pi_1| \# |\pi_2|}_{\gamma} \Gamma\{\emptyset\}$.

This leaves the case that both rules are active and neither is an axiom. We have:

$(\wedge - \vee)$:

$$\mathsf{cut}_{\sigma+1}\,\dfrac{\wedge\,\dfrac{\overset{\pi_{11}}{\Gamma\{B\}} \qquad \overset{\pi_{12}}{\Gamma\{C\}}}{\Gamma\{B \wedge C\}} \qquad \vee\,\dfrac{\overset{\pi_{21}}{\Gamma\{\bar{B}, \bar{C}\}}}{\Gamma\{\bar{B} \vee \bar{C}\}}}{\Gamma\{\emptyset\}} \qquad \rightsquigarrow$$

$$\mathsf{cut}_{\sigma}\,\dfrac{\overset{\pi_{11}}{\Gamma\{B\}} \qquad \mathsf{cut}_{\sigma}\,\dfrac{\mathsf{wk}\,\dfrac{\overset{\pi_{12}}{\Gamma\{C\}}}{\Gamma\{\bar{B}, C\}} \qquad \overset{\pi_{21}}{\Gamma\{\bar{B}, \bar{C}\}}}{\Gamma\{\bar{B}\}}}{\Gamma\{\emptyset\}} \qquad ,$$

where by weakening admissibility we get $\vdash^{|\pi_{12}|}_{\gamma} \Gamma\{\bar{B}, C\}$, and since $\sigma < \sigma{+}1 = \gamma$ we get $\vdash^{\alpha}_{\gamma} \Gamma\{\emptyset\}$ for $\alpha = max(|\pi_{11}|, max(|\pi_{12}|, |\pi_{21}|) + 1) + 1$. It is easy to check that $\alpha \leq |\pi_1| \# |\pi_2|$.

$(\Box_i - \Diamond_i)$:

$$\mathsf{cut}_{\sigma+1}\,\dfrac{\Box_i\,\dfrac{\overset{\pi_{11}}{\Gamma\{[\Delta]_i, [A]_i\}}}{\Gamma\{[\Delta]_i, \Box_i A\}} \qquad \Diamond_i\,\dfrac{\overset{\pi_{21}}{\Gamma\{[\Delta, \bar{A}]_i, \Diamond_i \bar{A}\}}}{\Gamma\{[\Delta]_i, \Diamond_i \bar{A}\}}}{\Gamma\{[\Delta]_i\}} \qquad \rightsquigarrow$$

$$\mathsf{cut}_{\sigma}\,\dfrac{\mathsf{ctr}\,\dfrac{\mathsf{wk}^2\,\dfrac{\overset{\pi_{11}}{\Gamma\{[\Delta]_i, [A]_i\}}}{\Gamma\{[\Delta, A]_i, [\Delta, A]_i\}}}{\Gamma\{[\Delta, A]_i\}} \qquad \mathsf{cut}_{\sigma+1}\,\dfrac{\mathsf{wk}, \Box_i\,\dfrac{\overset{\pi_{11}}{\Gamma\{[\Delta]_i, [A]_i\}}}{\Gamma\{[\Delta, \bar{A}]_i, \Box_i A\}} \qquad \overset{\pi_{21}}{\Gamma\{[\Delta, \bar{A}]_i, \Diamond_i \bar{A}\}}}{\Gamma\{[\Delta, \bar{A}]_i\}}}{\Gamma\{[\Delta]_i\}} \qquad ,$$

where the premises of the upper cut have been derived by use of weakening admissibility with depth $|\pi_{11}| + 1$ and $|\pi_{21}|$, the natural sum of which is smaller than $|\pi_1| \# |\pi_2|$. The induction hypothesis thus yields $\vdash^{(|\pi_{11}|+1) \# |\pi_{21}|}_{\gamma}$ $\Gamma\{[\Delta, \bar{A}]_i\}$ and since $\sigma < \sigma + 1 = \gamma$ we get $\vdash^{|\pi_1| \# |\pi_2|}_{\gamma}$ $\Gamma\{[\Delta]_i\}$ by the lower cut.

($\boxast - \diamondast$):

$$\mathsf{cut}_{\omega+\sigma}\,\dfrac{\boxast\,\dfrac{\vdots \quad \overset{\pi_{1k}}{\Gamma\{\Box^k A\}} \quad \vdots}{\Gamma\{\boxast A\}}\,{}_{1\leq k} \qquad \diamondast\,\dfrac{\overset{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\diamondast \bar{A}\}}}{\Gamma\{\emptyset\}} \qquad \rightsquigarrow$$

$$\mathsf{cut}_{\sigma+(j\cdot h)}\,\dfrac{\overset{\pi_{1j}}{\Gamma\{\Box^j A\}} \qquad \mathsf{cut}_{\omega+\sigma}\,\dfrac{\boxast\,\dfrac{\vdots \quad \mathsf{wk}\,\dfrac{\overset{\pi_{1k}}{\Gamma\{\Box^k A\}}}{\Gamma\{\Box^k A, \Diamond^j \bar{A}\}} \quad \vdots}{\Gamma\{\boxast A, \Diamond^j \bar{A}\}}\,{}_{1\leq k} \qquad \overset{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\Diamond^j \bar{A}\}}}{\Gamma\{\emptyset\}} \ ,$$

where the induction hypothesis applied on the upper cut gives us $\vdash^{|\pi_1| \# |\pi_{21}|}_{\gamma}$ $\Gamma\{\Diamond^j \bar{A}\}$ and since by Lemma 1 we have $\sigma + j \cdot h < \omega + \sigma = \gamma$ the lower cut yields $\vdash^{|\pi_1| \# |\pi_2|}_{\gamma}$ $\Gamma\{\emptyset\}$. $\Box$

From the reduction lemma we obtain the first and the second elimination lemma as usual, see for instance Pohlers [15, 16] or Schütte [18].

Lemma 6 (First Elimination Lemma)**.** *If* $\vdash^{\alpha}_{\gamma+1} \Gamma$ *then* $\vdash^{2^\alpha}_{\gamma} \Gamma$.

Proof. By induction on α and a case analysis on the last rule applied. Most cases are trivial, in case of a cut with rank γ we apply the induction hypothesis to both proofs of the premises of the cut and then apply the reduction lemma to obtain $\vdash^{2^{\alpha_0} \# 2^{\alpha_0}}_{\gamma} \Gamma$ for some $\alpha_0 < \alpha$ and thus $\vdash^{2^\alpha}_{\gamma} \Gamma$. $\Box$

Lemma 7 (Second Elimination Lemma)**.** *If* $\vdash^{\alpha}_{\beta+\omega^\gamma} \Gamma$ *then* $\vdash^{\varphi_\gamma \alpha}_{\beta} \Gamma$.

$$\begin{tikzcd}
 & \mathsf{G_C} + \mathsf{cut} \arrow[d, "\text{Thm 11}"] & \mathsf{G_C} \\
\mathsf{H_C} \arrow[r, "\text{Thm 18}"] & \mathsf{D_C} + \mathsf{cut} \arrow[r, "\text{Thm 8}"] & \mathsf{D_C} \arrow[u, "\text{Thm 16}"']
\end{tikzcd}$$

Figure 5: Overview of the various embeddings

Proof. By induction on γ with a subinduction on α. For $\gamma = 0$ this trivially follows from the first elimination lemma. Assume $\gamma > 0$. The non-trivial case is where the last rule in the given proof of Γ is a cut with a rank of β or greater. With $\Gamma = \Gamma\{\emptyset\}$ the proof is of the following form:

$$\mathsf{cut}_{<\beta+\omega^\gamma}\,\dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad\qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}\qquad .$$

Let $\alpha_0 = max(|\pi_1|, |\pi_2|)$. We apply the subinduction hypothesis on the subproofs of the cut and obtain $\vdash^{\varphi_\gamma(\alpha_0)}_{\beta} \Gamma\{A\}$ and $\vdash^{\varphi_\gamma(\alpha_0)}_{\beta} \Gamma\{\bar{A}\}$. Since $rk(A) < \beta + \omega^\gamma$ a quick calculation by case analysis on γ yields the existence of σ with $\sigma < \gamma$ and of n such that $rk(A) < \beta + \omega^\sigma \cdot n$. Thus, by a cut we obtain $\vdash^{\varphi_\gamma(\alpha_0)+1}_{\beta+\omega^\sigma\cdot n} \Gamma$. We apply the induction hypothesis n times to obtain $\vdash^{\varphi^n_\sigma(\varphi_\gamma(\alpha_0)+1)}_{\beta} \Gamma$, where φ^n_σ means φ_σ applied n times. Since $\varphi^n_\sigma(\varphi_\gamma(\alpha_0)+1) < \varphi_\gamma(\alpha)$ we have $\vdash^{\varphi_\gamma(\alpha)}_{\beta} \Gamma$. □

The cut-elimination theorem follows by iterated application of the second elimination lemma.

Theorem 8 (Cut-elimination for the deep system)**.**
If $\mathsf{D_C} \vdash^{\alpha}_{\omega\cdot n} \Gamma$ *then* $\mathsf{D_C} \vdash^{\varphi^n_1(\alpha)}_{0} \Gamma$.

4. Cut-Elimination for the Shallow System via the Deep System

In this section we give a cut-elimination procedure for the shallow system. To do so, we first embed the shallow system with cut into the deep system with cut, eliminate the cut there, and embed the cut-free deep system into the cut-free shallow system. Figure 5 gives an overview of the embeddings. We have seen the horizontal arrow on the right in the last section. Now we are going to see the vertical arrows. System $\mathsf{H_C}$ is a Hilbert system which we will see in the last section, together with the horizontal arrow on the left.

4.1. Embedding Shallow into Deep

This is the easy direction. We first define a notion of admissibility which is weaker than "depth-preserving": it allows the proof to grow by a finite amount.

Definition 9. A rule ρ is *finitely admissible* for a system $\mathcal{S}$ if for each instance of ρ with premises $\Gamma_1, \Gamma_2 \ldots$ and conclusion Δ there exists a finite ordinal n such that whenever $\mathcal{S} \mathrel{|\!\frac{\alpha}{\gamma}} \Gamma_i$ for all i then $\mathcal{S} \mathrel{|\!\frac{\alpha+n}{\gamma}} \Delta$.

Note that every perfectly admissible (that is, depth- and cut-rank-preserving admissible) rule is also finitely admissible: in that case the n in the above definition is zero. A finitary rule which is contained in a system is also finitely admissible for that system: in that case the n in the above definition is one. The cut rule, on the other hand, is generally not finitely admissible for (cut-free) infinitary systems.

Lemma 10. *The rule* $\mathsf{d}\,\dfrac{\Gamma\{[\diamondast A, \Delta]_i\}}{\Gamma\{\diamondast A, [\Delta]_i\}}$ *is finitely admissible for system* $\mathsf{D_C}$.

Proof. By induction on the depth of the proof of the premise. The only interesting case is the one with a $\diamondast$-rule:

$$
\diamondast\,\dfrac{\overset{\pi}{\Gamma\{[\diamondast A, \Diamond^k A, \Delta]_i\}}}{\mathsf{d}\,\dfrac{\Gamma\{[\diamondast A, \Delta]_i\}}{\Gamma\{\diamondast A, [\Delta]_i\}}}
\quad\rightsquigarrow\quad
\diamondast\,\dfrac{\mathsf{wk}, \vee^*\,\dfrac{\mathsf{wk}, \Diamond_i\,\dfrac{\mathsf{d}\,\dfrac{\overset{\pi}{\Gamma\{[\diamondast A, \Diamond^k A, \Delta]_i\}}}{\Gamma\{\diamondast A, [\Diamond^k A, \Delta]_i\}}}{\Gamma\{\diamondast A, \Diamond_i \Diamond^k A, [\Delta]_i\}}}{\Gamma\{\diamondast A, \Diamond^{k+1} A, [\Delta]_i\}}}{\Gamma\{\diamondast A, [\Delta]_i\}}
\quad ,
$$

where the instance of d shown on the right is removed by induction hypothesis. $\square$

Theorem 11 (Shallow into deep)**.** *If* $\mathsf{G_C} \mathrel{\vert\!\!\frac{\alpha}{\gamma}} \Gamma$ *then* $\mathsf{D_C} \mathrel{\vert\!\!\frac{\omega\cdot\alpha}{\gamma}} \Gamma$.

Proof. By induction on α and a case analysis on the last rule in the proof. Each rule of $\mathsf{G_C}$ except for the $\Box_i$-rule is a special case of its respective rule in $\mathsf{D_C}$. For the $\Box_i$-rule we have the following transformation:

$$
\Box_i\,\frac{\begin{array}{c}\pi\\ \Gamma, \diamondast\Delta, A\end{array}}{\Diamond_i\Gamma, \diamondast\Delta, \Box_i A, \Sigma}
\quad\rightsquigarrow\quad
\Box_i, \mathsf{wk}\,\frac{\mathsf{d}^*\,\dfrac{\mathsf{wk}^*, \Diamond_i^*\,\dfrac{\mathsf{nec}\,\dfrac{\begin{array}{c}\pi'\\ \Gamma, \diamondast\Delta, A\end{array}}{[\Gamma, \diamondast\Delta, A]_i}}{\Diamond_i\Gamma, [\diamondast\Delta, A]_i}}{\Diamond_i\Gamma, \diamondast\Delta, [A]_i}}{\Diamond_i\Gamma, \diamondast\Delta, \Box_i A, \Sigma}
\quad ,
$$

where π' is obtained by induction hypothesis. $\Box$

4.2. Embedding Deep into Shallow

The $\Box_i$-rule is the only rule in $\mathsf{G_C}$ which is not invertible. However, a slightly weaker property than invertibility holds, and we will need it in order to embed the deep system: if the conclusion of the $\Box_i$-rule is provable then either its premise is also provable or the conclusion is provable even after removing the main formula. This is stated more formally in the following lemma.

Lemma 12 (Quasi-invertibility of the $\Box_i$-rule)**.** *If there is a proof of the sequent* $\Box_i A, \Diamond_i\Gamma, \diamondast\Delta, \Sigma$ *in* $\mathsf{G_C}$ *then there is a proof of the same depth in* $\mathsf{G_C}$ *either of the sequent* $A, \Gamma, \diamondast\Delta$ *or of the sequent* $\Diamond_i\Gamma, \diamondast\Delta, \Sigma$.

Proof. By induction on the depth and a case analysis on the last rule in the given proof. If the endsequent is axiomatic then Σ is axiomatic and the second disjunct applies. If the last rule is the $\boxast$-rule then the proof is of the form

$$
\boxast\,\frac{\cdots \quad \begin{array}{c}\pi_k\\ \Box_i A, \Diamond_i\Gamma, \diamondast\Delta, \Sigma', \Box^k B\end{array} \quad \cdots\; {}_{1\leq k}}{\Box_i A, \Diamond_i\Gamma, \diamondast\Delta, \Sigma', \boxast B}
$$

We apply the induction hypothesis to each premise, with $\Sigma = \Sigma', \Box^k B$. If for some premise the first disjunct is true then we have a proof of $A, \Gamma, \diamondast\Delta$ and we have shown the first disjunct of our claim. If for all premises the second disjunct is true then for $1 \leq k < \omega$ we have proofs π'_k such that the following shows the second disjunct of our claim:

$$\boxast\ \dfrac{\vdots \quad \begin{array}{c}\pi'_k\\ \Diamond_i\Gamma, \diamondast\Delta, \Sigma', \Box^k B\end{array} \quad \vdots_{\ 1\leq k}}{\Diamond_i\Gamma, \diamondast\Delta, \Sigma', \boxast B}\ .$$

The cases for $\vee$ and $\wedge$ are similar.

If the last rule is the $\diamondast$-rule then the main formula is in $\diamondast\Delta$ or in Σ. Assume it is in $\diamondast\Delta$, the other case is similar. Then we have a proof of the form

$$\diamondast\ \dfrac{\begin{array}{c}\pi\\ \Box_i A, \Diamond_i\Gamma, \diamondast\Delta', \diamondast B, \Diamond B, \Sigma\end{array}}{\Box_i A, \Diamond_i\Gamma, \diamondast\Delta', \diamondast B, \Sigma}\ .$$

We apply the induction hypothesis. If the first disjunct is true then we have proved the first disjunct of our claim. If the second disjunct is true then we have a proof π' and the following shows the second disjunct of our claim:

$$\diamondast\ \dfrac{\begin{array}{c}\pi'\\ \Diamond_i\Gamma, \diamondast\Delta', \diamondast B, \Diamond B, \Sigma\end{array}}{\Diamond_i\Gamma, \diamondast\Delta', \diamondast B, \Sigma}\ .$$

If the last rule is the $\Box_j$-rule, then we distinguish two cases: 1) If $\Box_i A$ is the active formula then $j = i$ and the following transformation proves the first disjunct of our claim:

$$\Box_i\ \dfrac{\begin{array}{c}\pi\\ A, \Gamma, \diamondast\Delta\end{array}}{\Box_i A, \Diamond_i\Gamma, \diamondast\Delta, \Sigma} \quad \rightsquigarrow \quad \begin{array}{c}\pi\\ A, \Gamma, \diamondast\Delta\end{array}\ .$$

2) If $\Box_i A$ is not the active formula then the following transformation proves the second disjunct of our claim:

$$
\Box_j \frac{\begin{array}{c}\pi \\ \Gamma, \circledast\Delta, B\end{array}}{\Box_i A, \Diamond_j \Gamma, \circledast\Delta, \Sigma', \Box_j B}
\quad \rightsquigarrow \quad
\Box_j \frac{\begin{array}{c}\pi \\ \Gamma, \circledast\Delta, B\end{array}}{\Diamond_j \Gamma, \circledast\Delta, \Sigma', \Box_j B} \quad .
$$

$\Box$

In order to translate a derivation with deep rule applications into a derivation where only shallow rules are allowed we need a way of simulating the deep applicability. It turns out that, for certain shallow rules, if they are admissible for the shallow system, then their "deep version" is also admissible.

Definition 13 (Make a shallow rule deep)**.** Let $C\{\ \}$ be a formula context. If an instance of the rule ρ is shown on the left then an instance of the *rule* $C\{\rho\}$ is shown on the right:

$$
\rho \frac{\Gamma, A_1 \ \dots\ \Gamma, A_i\ \dots}{\Gamma, A} \qquad\qquad C\{\rho\} \frac{\Gamma, C\{A_1\}\ \dots\ \Gamma, C\{A_i\}\ \dots}{\Gamma, C\{A\}} \quad .
$$

For any formula context which only contains connectives from $\{\vee, \Box_1 \dots, \Box_h\}$ an instance of $C\{\rho\}$ is an instance of the *rule* $\check{\rho}$.

Lemma 14 (Deep applicability preserves finite admissibility)**.** *Let $C\{\ \}$ be a formula context which only contains connectives from $\{\vee, \Box_1 \dots, \Box_h\}$.*
(i) There is an n such that for all Γ we have $\mathsf{G_C} \vdash^{n}_{0} \Gamma, C\{p \vee \bar{p}\}$.
(ii) If a rule ρ is finitely admissible for $\mathsf{G_C}$ then $C\{\rho\}$ is also finitely admissible for system $\mathsf{G_C}$.
(iii) If a rule ρ is finitely admissible for $\mathsf{G_C}$ then $\check{\rho}$ is also finitely admissible for system $\mathsf{G_C}$.

Proof. Statement (iii) is immediate from (ii). Both (i) and (ii) are proved by induction on $C\{\ \}$. The case with $C\{\ \} = C_1\{\ \} \vee C_2$ is of course analogous to the case with $C\{\ \} = C_1 \vee C_2\{\ \}$ and is omitted. We first prove (i). The case that $C\{\ \}$ is empty is handled by an application of the $\vee$-rule. If

$C\{\ \} = C_1 \vee C_2\{\ \}$ or $C\{\ \} = \Box_i C_1\{\ \}$ then we obtain a proof respectively as follows:

$$
\vee\,\dfrac{\overset{\pi}{\Gamma, C_1, C_2\{p \vee \bar{p}\}}}{\Gamma, C_1 \vee C_2\{p \vee \bar{p}\}}
\qquad \text{or} \qquad
\Box_i\,\dfrac{\overset{\pi}{C_1\{p \vee \bar{p}\}}}{\Gamma, \Box_i C_1\{p \vee \bar{p}\}}
$$

where in both cases π exists by induction hypothesis. For statement (ii) the case that $C\{\ \}$ is empty is clear, so we assume that it is non-empty. If $C\{\ \} = C_1 \vee C_2\{\ \}$ then the following transformation proves our claim:

$$
C_1 \vee C_2\{\rho\}\,\dfrac{\vdots \quad \overset{\pi_k}{\Gamma, C_1 \vee C_2\{A_k\}} \quad \vdots}{\Gamma, C_1 \vee C_2\{A\}}
\quad \rightsquigarrow \quad
C_2\{\rho\}\,\dfrac{\vdots \quad \bar{\vee}\,\dfrac{\overset{\pi_k}{\Gamma, C_1 \vee C_2\{A_k\}}}{\Gamma, C_1, C_2\{A_k\}} \quad \vdots}{\vee\,\dfrac{\Gamma, C_1, C_2\{A\}}{\Gamma, C_1 \vee C_2\{A\}}}
$$

If $C\{\ \} = \Box_i C_1\{\ \}$ then we have the following situation:

$$
\Box_i C_1\{\rho\}\,\dfrac{\vdots \quad \overset{\pi_k}{\Gamma, \Box_i C_1\{A_k\}} \quad \vdots}{\Gamma, \Box_i C_1\{A\}} \quad .
$$

We apply quasi-invertibility of $\Box_i$, Lemma 12, to all π_k. Either this yields some proof π of Γ or for each k it yields a proof π'_k of some sequent $\Gamma', C_1\{A_k\}$. Then we can build either

$$
\mathsf{wk}\,\dfrac{\overset{\pi}{\Gamma}}{\Gamma, \Box_i C_1\{A\}}
\qquad \text{or} \qquad
C_1\{\rho\}\,\dfrac{\vdots \quad \overset{\pi'_k}{\Gamma', C_1\{A_k\}} \quad \vdots}{\Box_i\,\dfrac{\Gamma', C_1\{A\}}{\Gamma, \Box_i C_1\{A\}}} \quad ,
$$

$$\mathsf{g_c}\,\frac{\Gamma, A\vee B}{\Gamma, B\vee A} \qquad\qquad \mathsf{g_a}\,\frac{\Gamma,(A\vee B)\vee C}{\Gamma, A\vee(B\vee C)}$$

$$\mathsf{g_{ctr}}\,\frac{\Gamma,A\vee A}{\Gamma,A} \qquad \mathsf{g}_{\Diamond}\,\frac{\Gamma,\Box_i(A\vee B)}{\Gamma,\Diamond_i A,\Box_i B} \qquad \mathsf{g}_{\circledast}\,\frac{\Gamma,\Diamond^k A}{\Gamma,\circledast A} \text{ where } k\geq 1$$

Figure 6: Some glue

where in the second case $C_1\{\rho\}$ is finitely admissible by induction hypothesis. $\Box$

Lemma 15 (Some glue)**.** *The rules in Figure 6 are finitely admissible for system* $\mathsf{G_C}$*.*

Proof. The rules $\mathsf{g_c}, \mathsf{g_a}$ and $\mathsf{g_{ctr}}$ are easily seen to be finitely admissible by using invertibility of the $\vee$-rule. For the $\mathsf{g}_\Diamond$-rule we proceed by induction on the given proof of the premise and make a case analysis on the last rule in this proof. All cases are trivial except when this is the $\Box_i$-rule. We distinguish two cases: either 1) $\Box_i(A\vee B)$ is the active formula or 2) it is not. In the first case we have:

$$\mathsf{g}_\Diamond\,\dfrac{\Box_i\,\dfrac{\overset{\pi}{\circledast\Delta,\Lambda,A\vee B}}{\Sigma,\circledast\Delta,\Diamond_i\Lambda,\Box_i(A\vee B)}}{\Sigma,\circledast\Delta,\Diamond_i\Lambda,\Diamond_i A,\Box_i B} \quad\rightsquigarrow\quad \Box_i\,\dfrac{\bar{\vee}\,\dfrac{\overset{\pi}{\circledast\Delta,\Lambda,A\vee B}}{\circledast\Delta,\Lambda,A,B}}{\Sigma,\circledast\Delta,\Diamond_i\Lambda,\Diamond_i A,\Box_i B}$$

and in the second case we have the following:

$$\mathsf{g}_\Diamond\,\dfrac{\Box_i\,\dfrac{\overset{\pi}{C,\Gamma''}}{\Box_i C,\Gamma',\Box_i(A\vee B)}}{\Box_i C,\Gamma',\Diamond_i A,\Box_i B} \quad\rightsquigarrow\quad \Box_i\,\dfrac{\overset{\pi}{C,\Gamma''}}{\Box_i C,\Gamma',\Diamond_i A,\Box_i B} \quad .$$

For the $\mathsf{g}_{\diamondast}$-rule we proceed by induction on k and a subinduction on the depth of the given proof of the premise. For $k=1$ the $\mathsf{g}_{\diamondast}$-rule coincides with the $\diamondast$-rule plus a weakening, so we assume that we have a proof of $\Gamma, \Diamond^{k+1}A$. By invertibility of the $\vee$-rule we obtain a proof

$$\begin{array}{c} \pi \\ \Gamma, \Diamond_1\Diamond^k A, \ldots, \Diamond_h\Diamond^k A \end{array}$$

of the same depth. By induction on the depth of π and a case analysis on the last rule in π we now show that we have a proof of the same depth of $\Gamma, \diamondast A$. All cases are trivial except when the last rule is $\Box_i$. Then the following transformation:

$$\Box_i\,\dfrac{\begin{array}{c}\pi' \\ B, \Delta, \diamondast\Lambda, \Diamond^k A\end{array}}{\Box_i B, \Diamond_i\Delta, \diamondast\Lambda, \Sigma, \Diamond_1\Diamond^k A, \ldots, \Diamond_h\Diamond^k A} \quad \rightsquigarrow \quad \Box_i\,\dfrac{\mathsf{g}_{\diamondast}\,\dfrac{\begin{array}{c}\pi' \\ B, \Delta, \diamondast\Lambda, \Diamond^k A\end{array}}{B, \Delta, \diamondast\Lambda, \diamondast A}}{\Box_i B, \Diamond_i\Delta, \diamondast\Lambda, \Sigma, \diamondast A}$$

proves our claim, where the instance of the $\mathsf{g}_{\diamondast}$-rule on the right is finitely admissible by the outer induction hypothesis. $\Box$

For our translation from deep into shallow we translate nested sequents into formulas and thus fix an arbitrary order and association among elements of a sequent. The arbitrariness of this translation gets in the way, and we work around it as follows: we write $\mathsf{ac}\,\dfrac{A}{B}$ if the formula B can be derived from the formula A in $\{\check{\mathsf{g}}_{\mathsf{c}}, \check{\mathsf{g}}_{\mathsf{a}}\}$. Clearly, in that case A and B are equal modulo commutativity and associativity of disjunction. The converse is not the case. For example $\diamondast(C \vee D)$ can not be derived from $\diamondast(D \vee C)$ by ac, in general. Note that since $\check{\mathsf{g}}_{\mathsf{c}}$ and $\check{\mathsf{g}}_{\mathsf{a}}$ are finitely admissible for system $\mathsf{G_C}$, so is the rule ac.

Theorem 16 (Deep into shallow)**.**
If $\mathsf{D_C} \vdash^{\alpha}_{0} \Gamma$ *then we have* $\mathsf{G_C} \vdash^{\omega\cdot(\alpha+1)}_{0} \underline{\Gamma}_{\mathsf{F}}$.

Proof. By induction on α. If the endsequent of the given proof is of the form $\Gamma\{p, \bar{p}\}$, then we have

$$
\Gamma\{p, \bar{p}\} \quad \rightsquigarrow \quad \mathsf{ac}\,\dfrac{\dfrac{\pi}{\underline{\Gamma}_{\mathsf{F}}\{p \vee \bar{p}\}}}{\underline{\Gamma\{p, \bar{p}\}}_{\mathsf{F}}}
$$

where π is of finite depth by Lemma 14 and ac is finitely admissible by Lemma 15 and Lemma 14. If the last rule is the $\vee$-rule then an application of ac proves our claim. The case of the $\Box_i$-rule is trivial since the corresponding formula for the premise is the corresponding formula of the conclusion. For the $\boxast$-rule we apply the following transformation, where the π'_k are obtained by induction hypothesis:

$$
\boxast\,\dfrac{\cdots \quad \dfrac{\pi_k}{\Gamma\{\Box^k A\}} \quad \cdots}{\Gamma\{\boxast A\}}\;{\scriptstyle 1 \le k < \omega} \quad \rightsquigarrow \quad \underline{\Gamma}_{\mathsf{F}}\{\boxast\}\,\dfrac{\cdots \quad \mathsf{ac}\,\dfrac{\dfrac{\pi'_k}{\underline{\Gamma\{\Box^k A\}}_{\mathsf{F}}}}{\underline{\Gamma}_{\mathsf{F}}\{\Box^k A\}} \quad \cdots}{\mathsf{ac}\,\dfrac{\underline{\Gamma}_{\mathsf{F}}\{\boxast A\}}{\underline{\Gamma\{\boxast A\}}_{\mathsf{F}}}}\;{\scriptstyle 1 \le k < \omega}
$$

Let the depth of the proof on the left be β with $\beta \le \alpha$ and the depth of a proof π_k be β_k. Note that the depth of the ac-derivations both below and above the infinitary rule is bounded by a finite ordinal m because the context $\Gamma\{\;\}$ is finite. Then, by finite admissibility of the rule $\underline{\Gamma}_{\mathsf{F}}\{\boxast\}$ (Lemma 14) there is a finite ordinal n such that the proof on the right has the depth

$$
\begin{array}{rl}
 & \sup_k(|\pi'_k| + m + 1) + n + m \;<\; \sup_k(|\pi'_k|) + \omega \\
\le & \sup_k(\omega \cdot (\beta_k + 1)) + \omega \;=\; \omega \cdot \sup_k(\beta_k + 1) + \omega \\
= & \omega \cdot \beta + \omega \;=\; \omega \cdot (\beta + 1) \;\le\; \omega \cdot (\alpha + 1) \qquad .
\end{array}
$$

The case for the $\wedge$-rule is similar. For the $\Diamond_i$-rule we apply the following transformation, where π' is obtained by induction hypothesis and the bound on the depth is easy to check:

$$
\diamond_i \dfrac{\begin{array}{c}\pi \\ \Gamma\{\diamond_i A, [A,\Delta]_i\}\end{array}}{\Gamma\{\diamond_i A, [\Delta]_i\}}
\quad \rightsquigarrow \quad
\mathsf{ac}\,\dfrac{\underline{\Gamma}_{\mathsf{F}}\{\mathsf{g}_{\mathsf{ctr}} \vee \Box_i \underline{\Delta}_{\mathsf{F}}\}\,\dfrac{\mathsf{ac}\,\dfrac{\underline{\Gamma}_{\mathsf{F}}\{\diamond_i A \vee \mathsf{g}_{\diamond}\}\,\dfrac{\mathsf{ac}\,\dfrac{\begin{array}{c}\pi' \\ \underline{\Gamma\{\diamond_i A, [A,\Delta]_i\}}_{\mathsf{F}}\end{array}}{\underline{\Gamma}_{\mathsf{F}}\{\diamond_i A \vee \Box_i(A \vee \underline{\Delta}_{\mathsf{F}})\}}}{\underline{\Gamma}_{\mathsf{F}}\{\diamond_i A \vee (\diamond_i A \vee \Box_i \underline{\Delta}_{\mathsf{F}})\}}}{\underline{\Gamma}_{\mathsf{F}}\{(\diamond_i A \vee \diamond_i A) \vee \Box_i \underline{\Delta}_{\mathsf{F}}\}}}{\underline{\Gamma}_{\mathsf{F}}\{\diamond_i A \vee \Box_i \underline{\Delta}_{\mathsf{F}}\}}}{\underline{\Gamma\{\diamond_i A, [\Delta]_i\}}_{\mathsf{F}}}
\; .
$$

Note that here a rule like $C\{\rho \vee A\}$ means rule ρ applied in the context $C\{\{\ \} \vee A\}$, and is finitely admissible for $\mathsf{G_C}$ if is ρ is finitely admissible for $\mathsf{G_C}$, by Lemma 14.

The case for the $\circledast$-rule is similar. $\Box$

We can now state the cut-elimination theorem for the shallow system.

Theorem 17 (Cut-elimination for the shallow system)**.**
If $\mathsf{G_C} \vdash^{\alpha}_{\omega \cdot n} \Gamma$ *then* $\mathsf{G_C} \vdash^{\omega\cdot(\varphi_1^n(\omega\cdot\alpha)+1)}_{0} \Gamma$

5. An Upper Bound on the Depth of Proofs

The Hilbert system $\mathsf{H_C}$ is obtained from some Hilbert system for classical propositional logic by adding the axioms and rules shown in Figure 7. It is essentially the same as system K_h^C from the book [7], where also soundness and completeness are shown. We will now embed $\mathsf{H_C}$ into $\mathsf{D_C} + \mathsf{cut}$, keeping track of the proof depth and thus, via cut elimination for $\mathsf{D_C}$, establish an upper bound for proofs in $\mathsf{D_C}$. Via the embedding of the deep system into the shallow system, this bound also holds for the shallow system.

Theorem 18. *If* $\mathsf{H_C} \vdash A$ *then* $\mathsf{D_C} \vdash^{<\omega^2}_{\omega^2} A$.

Proof. The proof is by induction on the depth of the derivation in $\mathsf{H_C}$. If A is a propositional axiom of $\mathsf{H_C}$ then there is a finite derivation of A in the propositional part of system $\mathsf{D_C}$ such that all premises are instances of the general identity axiom. Thus we obtain $\mathsf{D_C} \vdash^{\omega\cdot m}_{0} A$ for some $m < \omega$ by admissibility of the general identity axiom (Lemma 4).

$$\text{(K)}\quad \Box_i A \wedge \Box_i(A \supset B) \supset \Box_i B \qquad\qquad \text{(CCL)}\quad \boxast A \supset (\Box A \wedge \Box \boxast A)$$

$$\text{(IND)}\ \frac{B \supset (\Box A \wedge \Box B)}{B \supset \boxast A} \qquad\qquad \text{(MP)}\ \frac{A \quad A \supset B}{B} \qquad\qquad \text{(NEC)}\ \frac{A}{\Box_i A}$$

Figure 7: System $\mathsf{H_C}$

If A is an instance of (K), then we obtain $\mathsf{D_C} \vdash^{\omega \cdot m}_{0} A$ for some $m < \omega$ from the following derivation and admissibility of the general identity axiom to take care of the premises.

$$
\vee^2\,\dfrac{\Box_i\,\dfrac{\Diamond_i\,\dfrac{\wedge\,\dfrac{\Diamond_i\,\dfrac{\Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), [B, A, \bar{A}]_i}{\Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), [B, A]_i} \qquad \Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), [B, \bar{B}]_i}{\Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), [B, A \wedge \bar{B}]_i}}{\Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), [B]_i}}{\Diamond_i \bar{A}, \Diamond_i(A \wedge \bar{B}), \Box_i B}}{\Box_i A \wedge \Box_i(A \supset B) \supset \Box_i B}
$$

If A is an instance of (CCL), then we obtain $\mathsf{D_C} \vdash^{\omega \cdot m}_{0} A$ for some $m < \omega$ from the following derivation and again admissibility of the general identity axiom to take care of the premises. An argument similar to the one used to derive the general identity axiom guarantees that all premises of the $\boxast$ rule are derivable with depth smaller than $rk(\boxast A)$.

$$
\vee\,\dfrac{\wedge\,\dfrac{\diamondast, \mathsf{wk}\,\dfrac{\Diamond \bar{A}, \Box A}{\diamondast \bar{A}, \Box A} \qquad \wedge\,\dfrac{\vdots \qquad \Box_i\,\dfrac{\boxast\,\dfrac{\vdots \qquad \diamondast, \mathsf{wk}\,\dfrac{\vee, \mathsf{wk}\,\dfrac{\Diamond_i, \mathsf{wk}\,\dfrac{[\Diamond^k \bar{A}, \Box^k A]_i}{\Diamond_i \Diamond^k \bar{A}, [\Box^k A]_i}}{\Diamond^{k+1} \bar{A}, [\Box^k A]_i}}{\diamondast \bar{A}, [\Box^k A]_i} \qquad \vdots_{\ 1 \le k < \omega}}{\diamondast \bar{A}, [\boxast A]_i}}{\diamondast \bar{A}, \Box_i \boxast A} \qquad \vdots_{\ 1 \le i \le h}}{\diamondast \bar{A}, \Box \boxast A}}{\diamondast \bar{A}, \Box A \wedge \Box \boxast A}}{\boxast A \supset (\Box A \wedge \Box \boxast A)}
$$

If the last rule in the derivation is an instance of (MP), then by the induction hypothesis there are $m_1, m_2, n_1, n_2 < \omega$ such that $\mathsf{D_C} \vdash^{\omega\cdot m_1}_{\omega\cdot n_1} A$ and $\mathsf{D_C} \vdash^{\omega\cdot m_2}_{\omega\cdot n_2} A \supset B$. Thus we get $\mathsf{D_C} \vdash^{\omega\cdot m_1}_{\omega\cdot n_1} A, B$ by weakening admissibility and $\mathsf{D_C} \vdash^{\omega\cdot m_2}_{\omega\cdot n_2} \bar{A}, B$ by invertibility. An application of cut yields $\mathsf{D_C} \vdash^{\omega\cdot m}_{\omega\cdot n} B$ for $m = max(m_1, m_2) + 1$ and $n = max(n_1, n_2, rk(A) + 1)$.

If the last rule in the derivation is an instance of (NEC), then the claim follows from the induction hypothesis, the fact that nec is cut-rank- and depth-preserving admissible, and an application of $\Box_i$.

If the last rule in the derivation is an instance of (IND), then by the induction hypothesis there are $m_1, n_1 < \omega$ such that $\mathsf{D_C} \vdash^{\omega\cdot m_1}_{\omega\cdot n_1} B \supset (\Box A \wedge \Box B)$. Then by invertibility of the $\wedge$- and $\vee$-rules we obtain

$$1)\ \mathsf{D_C} \vdash^{\omega\cdot m_1}_{\omega\cdot n_1} \bar{B}, \Box B \qquad \text{and} \qquad 2)\ \mathsf{D_C} \vdash^{\omega\cdot m_1}_{\omega\cdot n_1} \bar{B}, \Box A.$$

Let n_2 be such that $rk(\Box B) < \omega \cdot n_2$. We set $n = max(n_1, n_2)$. By induction on k we show that for all $k \geq 1$ there is an $m_2 < \omega$ such that $\mathsf{D_C} \vdash^{\omega\cdot m_1 + m_2}_{\omega\cdot n} \bar{B}, \Box^k A$. The case $k = 1$ is given by 2) and the induction step is as follows:

$$\dfrac{\bar{B}, \Box B \qquad \dfrac{\vdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\bar{B}, \Box^k A}{[\bar{B}, \Box^k A]_i}\mathsf{nec}}{\Diamond_i \bar{B}, [\Box^k A]_i}\Diamond_i, \mathsf{wk}}{\Diamond_i \bar{B}, \Box_i \Box^k A}\Box_i}{\Diamond \bar{B}, \Box_i \Box^k A}\vee, \mathsf{wk} \qquad \vdots \quad {}_{1\leq i\leq h}}{\Diamond \bar{B}, \Box^{k+1} A}\wedge}{\bar{B}, \Box^{k+1} A}\mathsf{cut} \quad ,$$

where the premise on the left is 1) and the premise on the right follows by induction hypothesis. The claim follows by applications of $\circledast$ and $\vee$. $\Box$

The embedding of the Hilbert system into the deep sequent system together with the cut-elimination theorem for the deep system gives us the following upper bounds on the depth of proofs in the cut-free systems.

Theorem 19 (Upper bounds)**.** *If A is a valid formula then*
(i) $\mathsf{D_C} \vdash^{<\varphi_2 0}_{0} A$, *and*
(ii) $\mathsf{G_C} \vdash^{<\varphi_2 0}_{0} A$.

Proof. If A is valid then by completeness of $\mathsf{H_C}$ we have $\mathsf{H_C} \vdash A$ and by the embedding of the Hilbert system into the deep sequent system there are natural numbers m, n such that $\mathsf{D_C} \vdash^{\omega \cdot m}_{\omega \cdot n} A$. By the cut elimination theorem for the deep sequent system we obtain $\mathsf{D_C} \vdash^{\varphi_1^n(\omega \cdot m)}_{0} A$. We know $\varphi_{\beta_1}\gamma_1 < \varphi_{\beta_2}\gamma_2$ if $\beta_1 < \beta_2$ and $\gamma_1 < \varphi_{\beta_2}\gamma_2$. Thus $\mathsf{D_C} \vdash^{<\varphi_2 0}_{0} A$. For (ii) by the embedding of the deep system into the shallow system it suffices to check that for $\alpha < \varphi_2 0$ we have $\omega \cdot (\alpha + 1) < \varphi_2 0$. ∎

6. Conclusion

We have introduced an deep sequent system for common knowledge which, in contrast to the shallow system by Alberucci and Jäger, admits a syntactic cut-elimination procedure. We have shown this cut-elimination procedure, and, via embedding the two systems into each other, have also provided a cut-elimination procedure for the shallow system. We embedded a Hilbert style system and obtained $\varphi_2 0$ as upper bound on the depth of cut-free proofs for both sequent systems.

We have looked at common knowledge based on the least normal modal logic. In a sense, "common belief" would be a better name. We believe that our approach is independent of the particular axiomatisation of knowledge. The modal logic S5 is widely seen as more adequate for knowledge than the modal logic K. Contrary to shallow sequents, nested sequents can easily handle S5. So it is easy to design a system for S5-based common knowledge. Generalising contexts to allow two holes, the single rule rule to add would be

$$\mathsf{S5}\ \frac{\Gamma\{\Diamond A\}\{A\}}{\Gamma\{\Diamond A\}\{\emptyset\}} \quad .$$

Of course there are also more speculative questions. What is the mathematical meaning of the upper bound on the depth of cut-free proofs? Is there a kind of boundedness lemma in modal logic similar to the one used in the analysis of set theories and second order arithmetic? Is $\varphi_2 0$ the best possible upper bound on the depth of proofs? What would be the equivalent of a well-ordering proof in modal logic? And finally, how could one syntactically eliminate cuts in a finitary system?

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