Yet Another Proof Theory for Modal Logic

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PRELIMINARIES

Formulas : $A ::= p \mid \bar{p} \mid A \vee A \mid A \wedge A \mid \Diamond A \mid \Box A$

Models : $\mathcal{M} = (S, R, V)$

S: set of states

$R \subseteq S \times S$

$V : \mathrm{Prop} \longrightarrow \mathcal{P}(S)$

(S, R) : **frame**

Entailment :

$\mathcal{M}, s \vDash p \iff V(p) \ni s$

$\vdots$

$\mathcal{M}, s \vDash \Diamond A \iff \exists t$ with sRt. $t \vDash A$

$\mathcal{M}, s \vDash \Box A \iff \forall t$ with sRt. $t \vDash A$

A is **valid** iff $\forall \mathcal{M} \forall s$ $\mathcal{M}, s \vDash A$.

A is **valid for a class of frames F** iff it is valid in all models with frames from F.

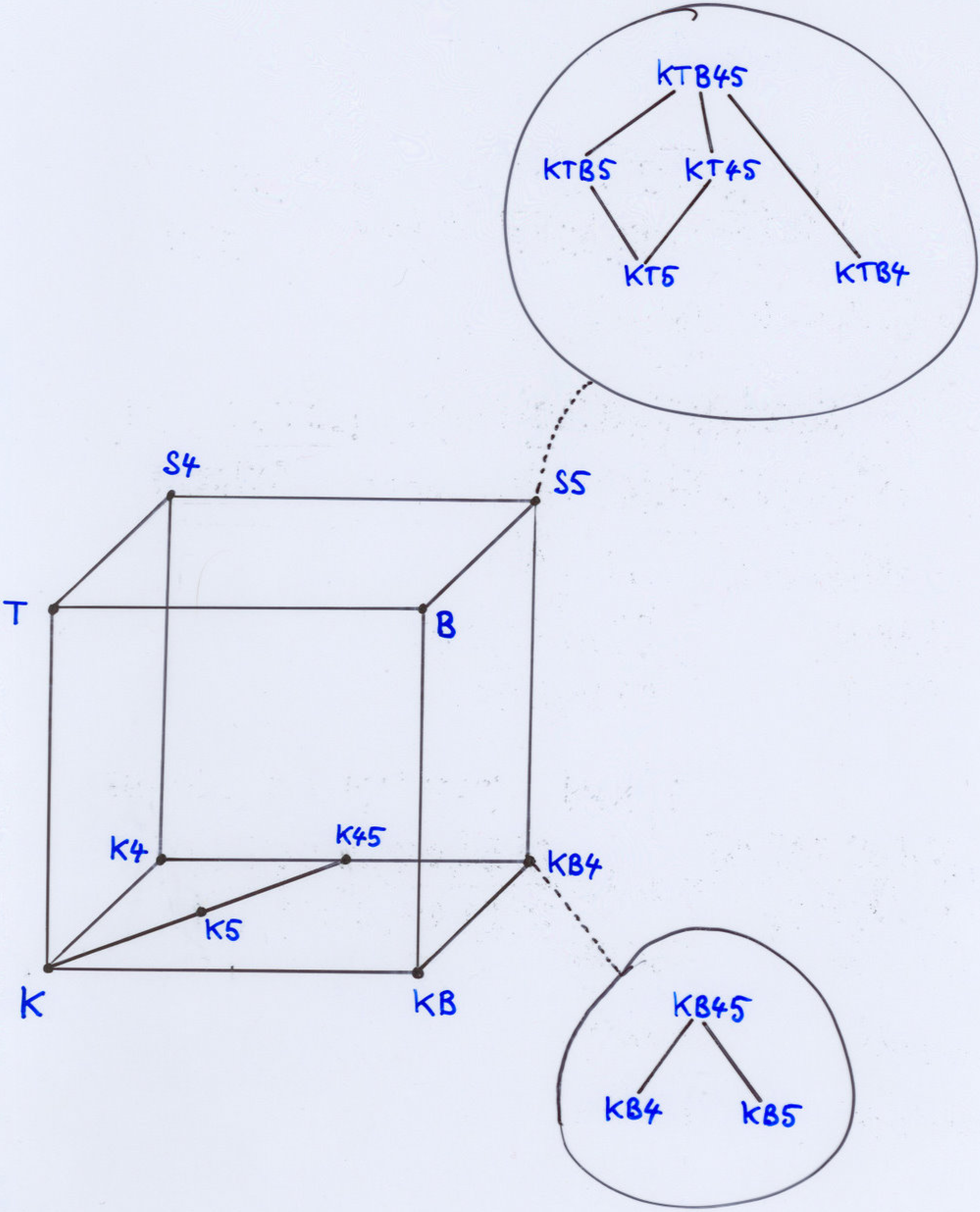
Hilbert Axiomatisations

The modal logic K, the logic of all frames is axiomatised in system HK:

1) classical propositional Hilbert system
2) axiom k: $\Box(A \supset B) \supset (\Box A \supset \Box B)$
3) inference rule Nec: $\dfrac{A}{\Box A}$

Some Axioms

t: $\Box A \supset A$	reflexive
4: $\Box A \supset \Box\Box A$	transitive
b: $A \supset \Box\Diamond A$	symmetric
5: $\Diamond A \supset \Box\Diamond A$	euclidean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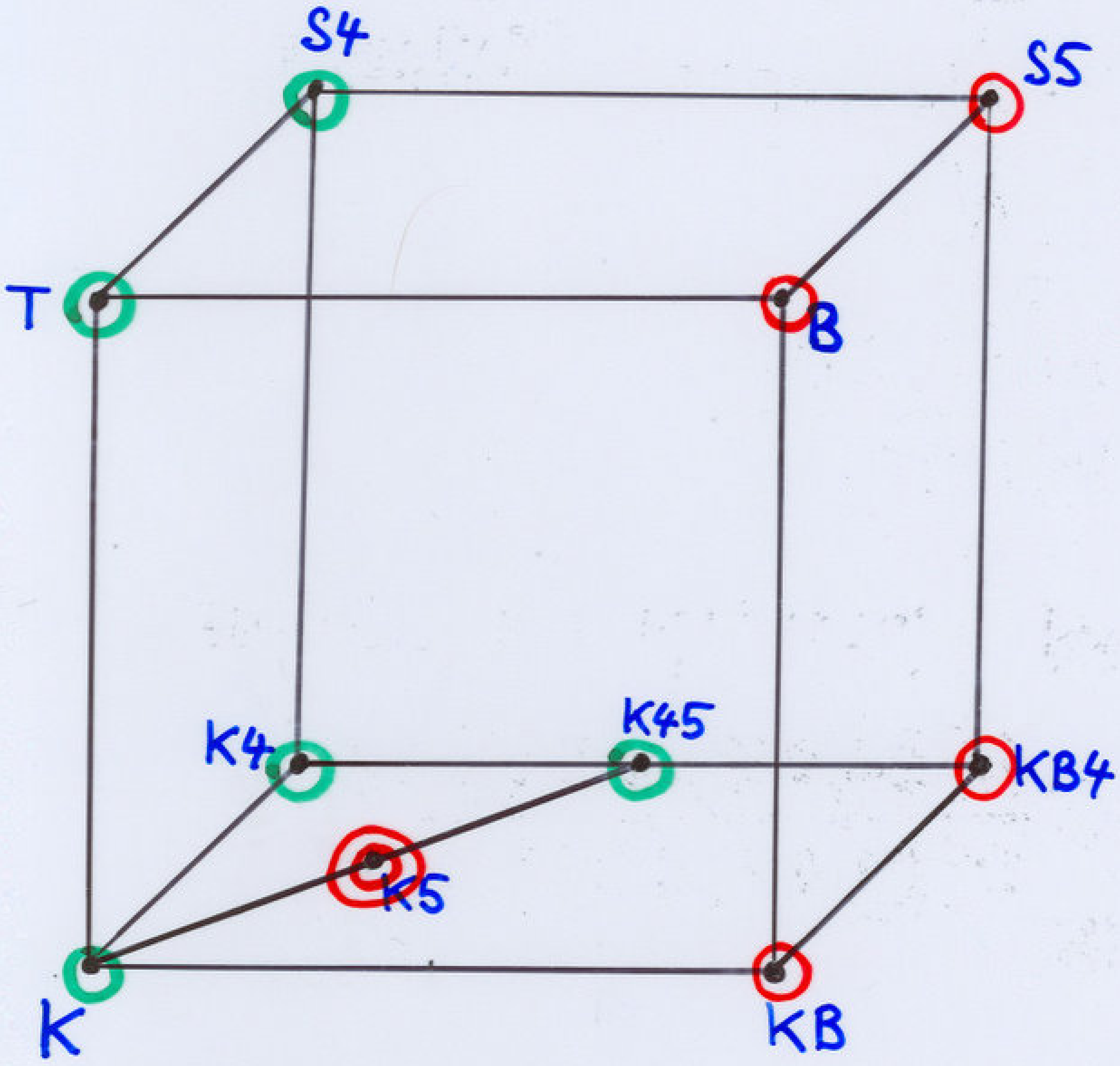
Frame Correspondence Theorem

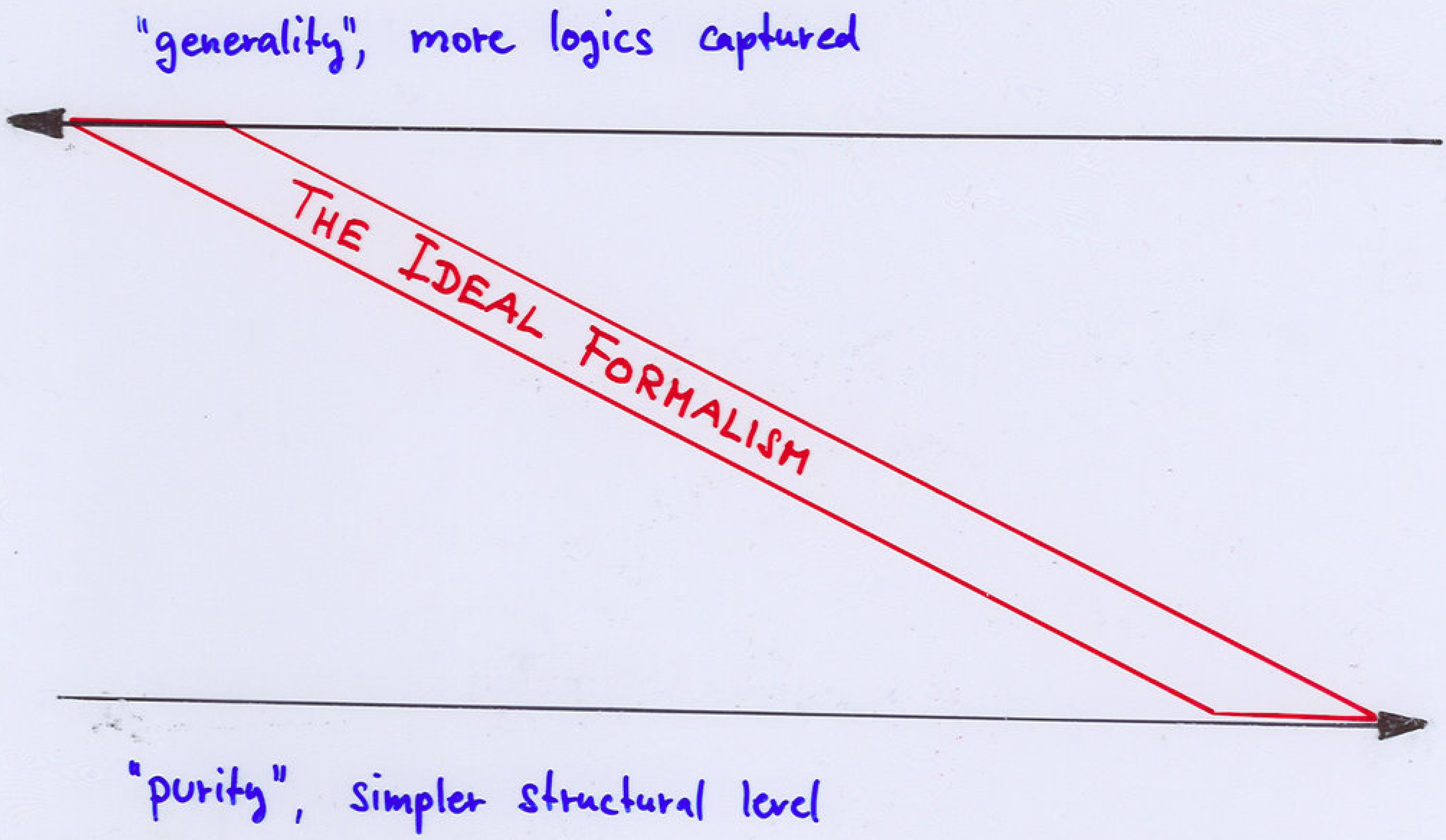
For each $X \subseteq \{t, b, 4, 5\}$, for each A

$HK + X \vdash A$ iff A is valid in all X-frames.

CUT-FREE SYSTEMS IN THE SEQUENT CALCULUS

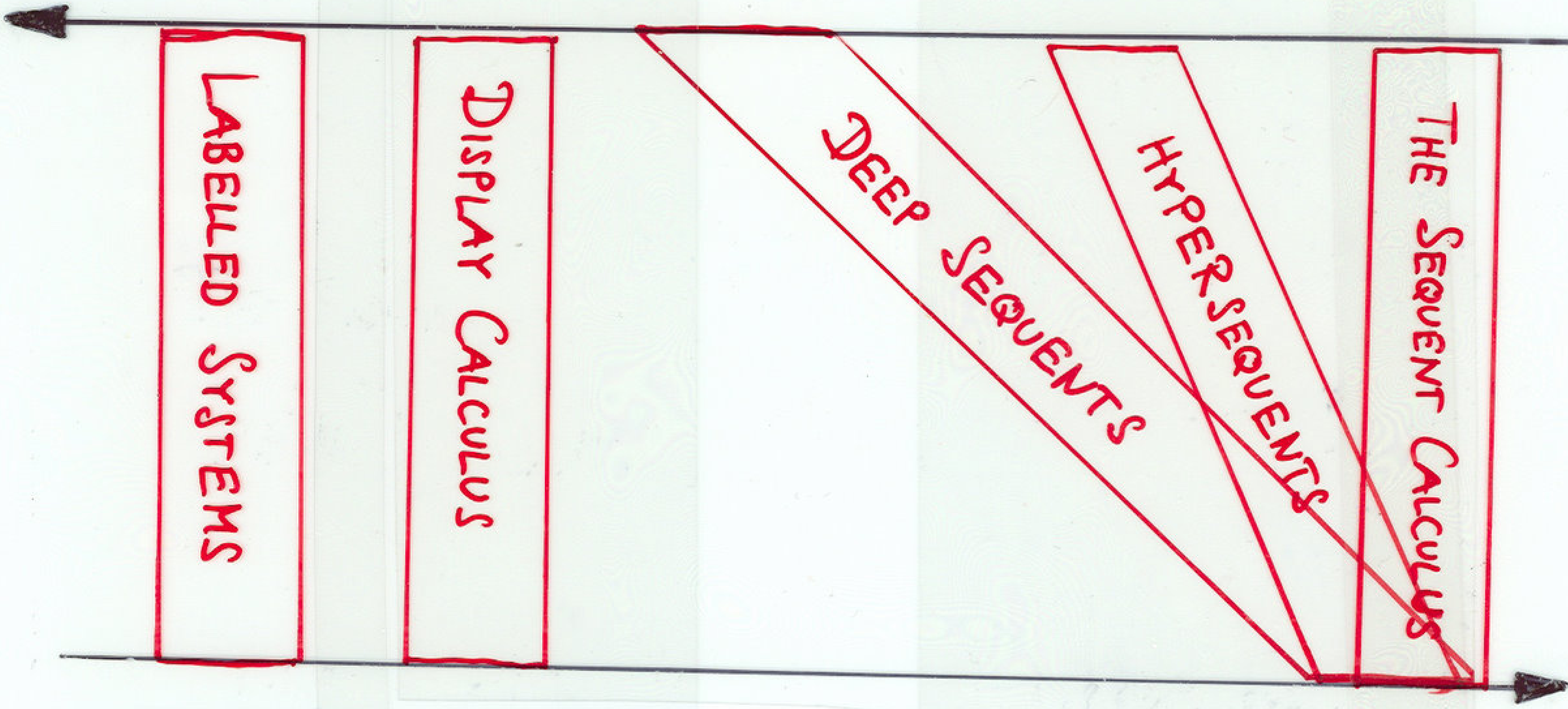


BEYOND THE SEQUENT CALCULUS



BEYOND THE SEQUENT CALCULUS

"generality", more logics captured



"purity", simpler structural level

DEEP SEQUENTS

Def A **sequent** is a multiset of formulas and boxed sequents.

A **boxed sequent** is an expression $[\Gamma]$ where Γ is a sequent.

Example

$$A, [B], [C, [D,E]] \quad \text{is a sequent}$$

induces a tree:

$$\begin{tikzcd}
 & \{A\} \arrow[dl] \arrow[dr] & \\
\{B\} & & \{C\} \arrow[d] \\
 & & \{D,E\}
\end{tikzcd}$$

Def A **sequent context**, denoted $\Gamma\{\}$ is a sequent with exactly one occurrence of the symbol $\{\}$.

The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\}$ by Δ in $\Gamma\{\}$.

THE MODAL SYSTEMS

$$\Gamma\{a,\bar{a}\} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\ \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}$$

$$\Box\ \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad k\ \frac{\Gamma\{\Diamond A, [\Delta, A]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

SYSTEM K

$$t\ \frac{\Gamma\{\Diamond A, A\}}{\Gamma\{\Diamond A\}} \qquad 4\ \frac{\Gamma\{\Diamond A, [\Delta, \Diamond A]\}}{\Gamma\{\Diamond A, [\Delta]\}}$$

$$b\ \frac{\Gamma\{[\Delta, \Diamond A], A\}}{\Gamma\{[\Delta, \Diamond A]\}} \qquad 5\ \frac{\Gamma\{[\Delta, \Diamond A], \Lambda\{\Diamond A\}\}}{\Gamma\{[\Delta, \Diamond A], \Lambda\{\emptyset\}\}}$$

Some Syntactic Remarks

1) $\mathsf{N}\dfrac{\Gamma}{[\Gamma]}$, $\mathsf{w}\dfrac{\Gamma\{\Delta\}}{\Gamma\{\Delta,\Delta'\}}$, $\mathsf{C}\dfrac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$

are depth-preserving admissible for $K+X$.

2) For each system $K+X$ all rules are depth-preserving invertible.

3) Cut-free sequent calculi for $K, T, S4$ can be embedded into K, $K+\{t\}$, $K+\{t,4\}$. (probably more)

4) There is a syntactic cut elimination procedure for K and $K+\{t\}$.

Cut is $\bowtie\dfrac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$

THEOREM

For each modal logic of the cube there is an $X \subseteq \{t, b, 4, 5\}$ s.t. $K + X$ is complete for this logic.

Completeness Proof

- prove the contraposition:

$$K + X \nvdash A \implies \exists \mathcal{M} \text{ on an } X\text{-frame s.t. } \mathcal{M} \not\models A$$

- build a proof-tree on A according to some strategy, s.t. for each leaf its sequent has only leafs that are either 1) cyclic or 2) finished.

- by assumption this fails, so we have a non-ax. sequent Γ with all leaves cyclic or finished

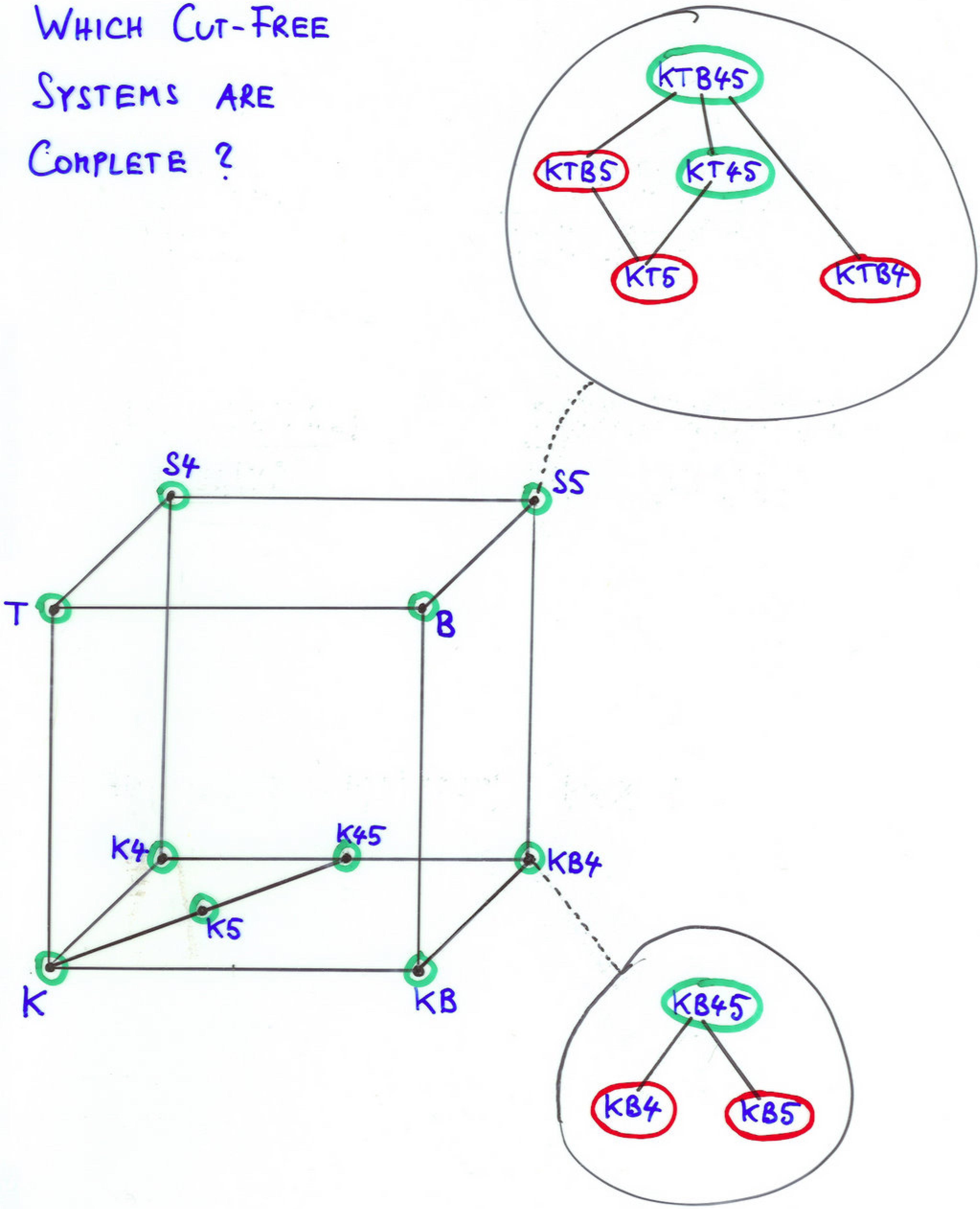
-  $=: (S, R)$

- $V(p) := \{\Delta \in S \mid \bar{p} \in \Delta\}$ $\quad \mathcal{M} = (S, R^{*}, V)$

- to prove: $\mathcal{M}, \Gamma \not\models \Gamma$

  to prove: $\forall B, \forall \Delta, \Delta'$ with $\Delta R^{*} \Delta' : \Diamond B \in \Delta \implies B \in \Delta'$

Which Cut-Free Systems are Complete ?



Towards Modularity

$$t^\bullet \ \frac{\Gamma\{\Sigma,[\Sigma]\}}{\Gamma\{\Sigma\}} \qquad 4^\bullet \ \frac{\Gamma\{[\Delta_1,[\Sigma,\Delta_2]],[\Sigma]\}}{\Gamma\{[\Delta_1,[\Sigma,\Delta_2]]\}}$$

$$b^\bullet \ \frac{\Gamma\{\Sigma,[\Delta,[\Sigma]]\}}{\Gamma\{\Sigma,[\Delta]\}} \qquad 5^\bullet \ \frac{\Gamma\{[\Delta_1,[\Sigma]],[\Delta_2,\Sigma]\}}{\Gamma\{[\Delta_1],[\Delta_2,\Sigma]\}}$$

Conjecture

For each $X \subseteq \{t^\bullet, b^\bullet, 4^\bullet, 5^\bullet\}$ $K+X$ is complete for the class of X-frames.