

WHAT'S A CUT-FREE DERIVATION IN  
THE CALCULUS OF STRUCTURES ?

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## FORMULAS

$$A ::= p \mid \bar{p} \mid t \mid f \mid [A, A] \mid (A, A)$$

- Negation  $\bar{A}$  defined via De Morgan
- Formulas are considered equal modulo associativity and commutativity of conjunction and disjunction and  $t$  being a unit for conjunction and  $f$  a unit for disjunction.

## INFERENCE RULES

$$i\downarrow \frac{S\{t\}}{S[A, \bar{A}]}$$

$$i\uparrow \frac{S(A, \bar{A})}{S\{f\}}$$

# SYSTEM SKS

$$i \downarrow \frac{S\{t\}}{S[A, \bar{A}]}$$

$$i \uparrow \frac{S(A, \bar{A})}{S\{t\}}$$

$$s \frac{S(A, [B, C])}{S[(A, B), C]}$$

$$c \downarrow \frac{S[A, A]}{S\{A\}}$$

$$c \uparrow \frac{S\{A\}}{S(A, A)}$$

$$w \downarrow \frac{S\{t\}}{S\{A\}}$$

$$w \uparrow \frac{S\{A\}}{S\{t\}}$$

# DERIVATIONS

$$\begin{array}{l}
 w \uparrow \\
 \frac{[(A, B), A]}{=} \\
 \frac{[(A, \dagger), A]}{=} \\
 c \downarrow \\
 \frac{[A, A]}{A}
 \end{array}$$

$$\begin{array}{l}
 c \uparrow \\
 \frac{\bar{A}}{=} \\
 \frac{(\bar{A}, \bar{A})}{=} \\
 w \downarrow \\
 \frac{([\bar{A}, \dagger], \bar{A})}{([\bar{A}, \bar{B}], \bar{A})}
 \end{array}$$

Derivation from A to B  
using rules from  $\mathcal{J}$  :

$$\begin{array}{l}
 A \\
 \parallel \mathcal{J} \\
 B
 \end{array}$$

Proof of A using rules  
from  $\mathcal{J}$  :

$$\begin{array}{l}
 \epsilon \\
 \parallel \mathcal{J} \\
 A
 \end{array}
 \quad \text{or} \quad
 \begin{array}{l}
 \top \\
 \parallel \mathcal{J} \\
 A
 \end{array}$$

# FIRST NOTION OF CUT

"ASYMMETRIC"

Theorem (Cut Elimination I)

For every proof  $\frac{\Pi \downarrow, \uparrow}{A}$  there is a proof  $\frac{\Pi \downarrow}{A}$ .

cut = up-rule

Is that the right notion?

How do we recognise that something is the "right" notion?

e.g. Is the cut elimination theorem powerful?

# DESIRABLE IMMEDIATE CONSEQUENCES

## OF CUT ELIMINATION IN THE SEQUENT CALCULUS

- consistency
- decidability
- Herbrand's Theorem
- Craig Interpolation & Definability

in CoS ?

(✓)

✓

✓

✗

## SECOND NOTION OF CUT

"SYMMETRIC"

### Theorem (Cut Elimination II)

For each derivation  $\begin{array}{c} A \\ \parallel \uparrow, \downarrow \\ B \end{array}$  there is a derivation

$\begin{array}{c} A \\ \parallel \uparrow \\ C \\ \parallel \downarrow \\ B \end{array}$ .

Cut = up-rule below down-rule  
= down-rule above up-rule

### Corollary (Cut Elimination I)

For each proof  $\begin{array}{c} \Pi \uparrow \\ B \end{array}$  there is a proof  $\begin{array}{c} \Pi \downarrow \\ B \end{array}$ .

### Corollary (Craig Interpolation)

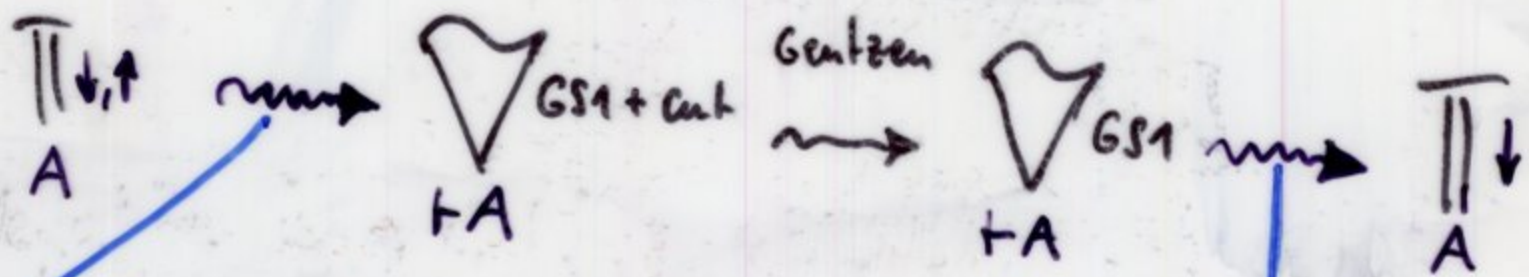
If  $A \Rightarrow B$  then  $\exists C$ .  $A \Rightarrow C \wedge$   
 $C \Rightarrow B \wedge$   
 $\ell(C) \subseteq \ell(A) \cap \ell(B)$ .

# CUT ELIMINATION I VIA SEQUENT CALCULUS

## Theorem

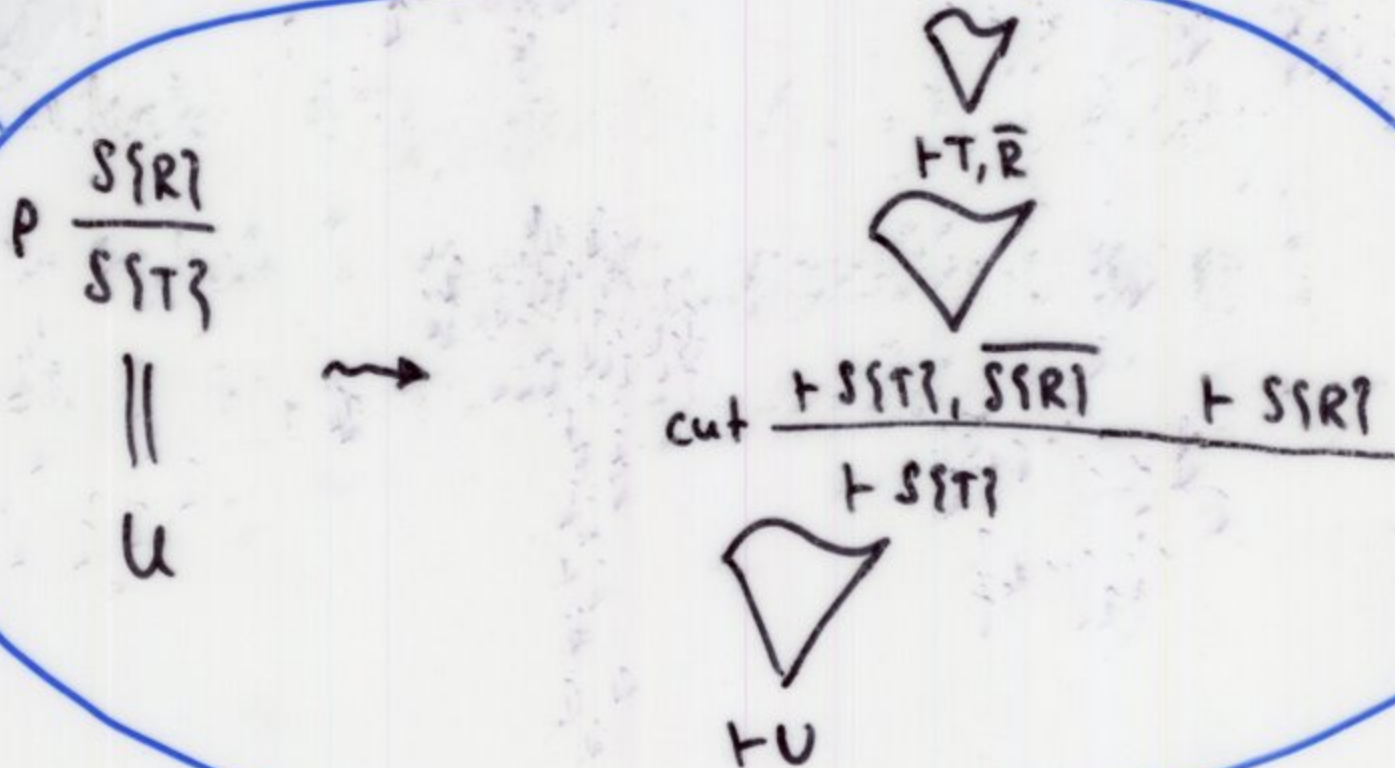
For each proof  $\frac{\Pi \uparrow, \downarrow}{A}$  there is one  $\frac{\Pi \downarrow}{A}$ .

## Proof



example:

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \wedge B} \rightsquigarrow \frac{\begin{array}{c} \frac{([\Gamma, A], [\Delta, B])}{[\Gamma, (A, [\Delta, B])]} \\ S \\ \frac{[\Gamma, \Delta, (A, B)]}{S} \end{array}}$$



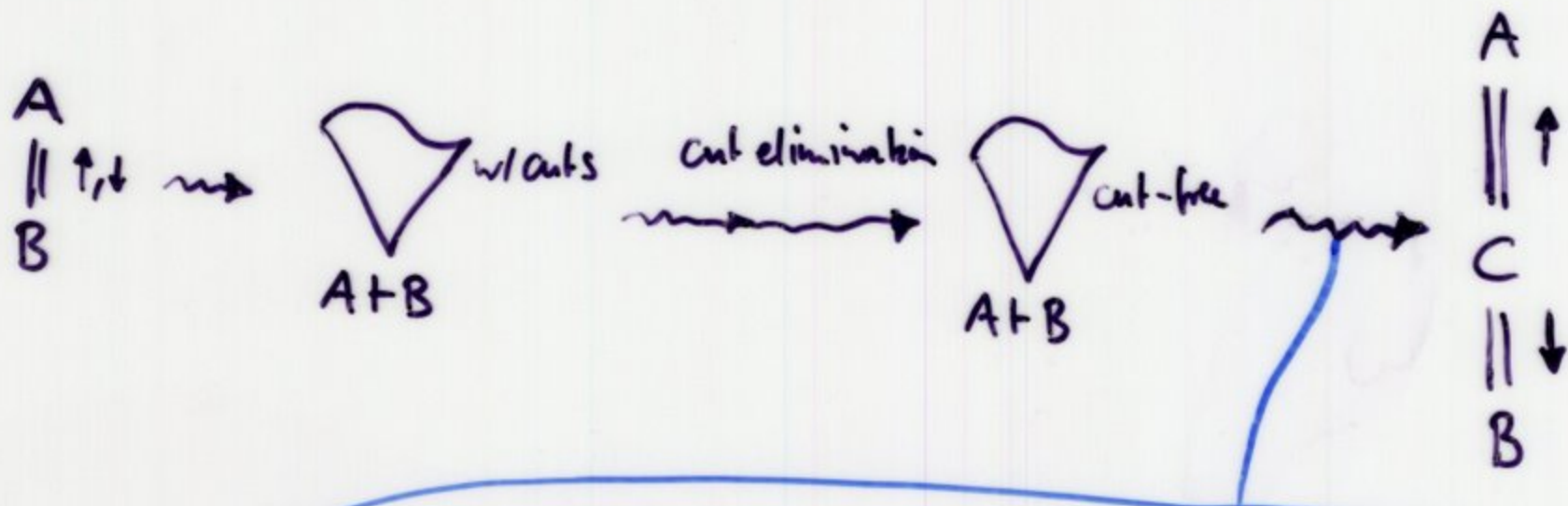
□



# PROOF OF CUT ELIMINATION II

use two-sided sequent calculus like LK but:

- formulas restricted to negation normal form
- no negation rules
- axioms:  $a \vdash a$   
 $a, \bar{a} \vdash$   
 $\vdash a, \bar{a}$



$$a \vdash a \rightsquigarrow a$$

$$a, \bar{a} \vdash \rightsquigarrow \text{if } \frac{(a, \bar{a})}{f}$$

$$\vdash a, \bar{a} \rightsquigarrow \text{if } \frac{\vdash}{[a, \bar{a}]}$$

$$\begin{array}{c} \text{cut} \\ \frac{\Phi, A, A \vdash \Psi}{\Phi, A \vdash \Psi} \\ \text{CL} \end{array} \rightsquigarrow \begin{array}{c} \text{c} \uparrow \\ \frac{\Phi, A}{(\Phi, A, A)} \\ \parallel \uparrow \\ C \\ \parallel \downarrow \\ \Psi \end{array}$$

$$\begin{array}{c} \text{cut} \\ \frac{\Phi \vdash A, A, \Psi}{\Phi \vdash A, \Psi} \end{array} \rightsquigarrow \begin{array}{c} \Phi \\ \parallel \uparrow \\ C \\ \parallel \downarrow \\ \frac{[A, A, \Psi]}{[A, \Psi]} \end{array}$$

$$\wedge_R \frac{\begin{array}{c} \text{▽} \\ \Phi \vdash A, \Psi \end{array} \quad \begin{array}{c} \text{▽} \\ \Phi' \vdash B, \Psi' \end{array}}{\Phi, \Phi' \vdash A \wedge B, \Psi, \Psi'}$$

→

$$\begin{array}{c} (\Phi, \Phi') \\ \parallel \uparrow \parallel \uparrow \\ (C, C') \\ \parallel \downarrow \parallel \downarrow \\ \frac{([A, \Psi] [B, \Psi'])}{s^2} \\ [(A, B), \Psi, \Psi'] \end{array}$$

$$\text{Cut} \frac{\begin{array}{c} \text{▽} \\ \Phi \vdash A, \Psi \end{array} \quad \begin{array}{c} \text{▽} \\ \Phi' \vdash A, \Psi' \end{array}}{\Phi, \Phi' \vdash \Psi, \Psi'}$$

→

$$\begin{array}{c} (\Phi', \Phi) \\ \parallel \uparrow \\ (\Phi', C) \\ \parallel \downarrow \\ \frac{(\Phi', [A, \Psi])}{s} \\ [\Psi, (\Phi', A)] \\ \parallel \uparrow \\ [\Psi, C'] \\ \parallel \downarrow \\ [\Psi, \Psi'] \end{array}$$

PROBLEM 1 Find an internal cut elimination II procedure, i.e. without reference to sequent calculus.

PROBLEM 2 Prove cut elimination II for logics without cut-free sequent calculus, like BV, or modal logics like B, K5, S5.

PROBLEM 3 Computational interpretation, intuitionistic logic