What's a Cut-Free Derivation in the Calculus of Structures ?

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FORMULAS

$$A ::= p \mid \bar{p} \mid t \mid f \mid [A, A] \mid (A, A)$$

- Negation $\bar{A}$ defined via De Morgan
- Formulas are considered equal modulo associativity and commutativity of conjunction and disjunction and t being a unit for conjunction and f a unit for disjunction.

INFERENCE RULES

$$\text{i}{\downarrow}\ \frac{S\{t\}}{S[A, \bar{A}]} \qquad \text{i}{\uparrow}\ \frac{S(A, \bar{A})}{S\{f\}}$$

System SKS

$$\mathsf{i}{\downarrow}\,\frac{S\{t\}}{S[A,\bar{A}]} \qquad \mathsf{i}{\uparrow}\,\frac{S(A,\bar{A})}{S\{f\}}$$

$$\mathsf{s}\,\frac{S(A,[B,C])}{S[(A,B),C]}$$

$$\mathsf{c}{\downarrow}\,\frac{S[A,A]}{S\{A\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{A\}}{S(A,A)}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{f\}}{S\{A\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{A\}}{S\{t\}}$$

Derivations

$$\dfrac{\dfrac{\dfrac{[(A,B),A]}{[(A,t),A]}\,\mathsf{w}\uparrow}{[A,A]}\,=}{A}\,\mathsf{c}\downarrow \qquad\qquad \dfrac{\dfrac{\dfrac{\bar{A}}{(\bar{A},\bar{A})}\,\mathsf{c}\uparrow}{([\bar{A},f],\bar{A})}\,=}{([\bar{A},\bar{B}],\bar{A})}\,\mathsf{w}\downarrow$$

Derivation from A to B using rules from $\mathcal{S}$: $\begin{array}{c} A \\ \| \, \mathcal{S} \\ B \end{array}$

Proof of A using rules from $\mathcal{S}$: $\begin{array}{c} t \\ \| \, \mathcal{S} \\ A \end{array}$ or $\begin{array}{c} \prod \mathcal{S} \\ A \end{array}$

First Notion of Cut

"Asymmetric"

Theorem (Cut Elimination I)

For every proof $\begin{array}{c}\Vert^{\downarrow,\uparrow} \\ A\end{array}$ there is a proof $\begin{array}{c}\Vert^{\downarrow} \\ A\end{array}$.

$$\text{cut} = \text{up-rule}$$

Is that the right notion?

How do we recognise that something is the "right" notion?

e.g. Is the cut elimination theorem powerful?

Desirable Immediate Consequences of Cut Elimination in the Sequent Calculus

	in CoS ?
• consistency	(✓)
• decidability	✓
• Herbrand's Theorem	✓
• Craig Interpolation & Definability	✗

Second Notion of Cut
"Symmetric"

Theorem (Cut Elimination II)

For each derivation $\begin{array}{c} A \\ \Vert\,{\uparrow,\downarrow} \\ B \end{array}$ there is a derivation $\begin{array}{c} A \\ \Vert\,{\uparrow} \\ C \\ \Vert\,{\downarrow} \\ B \end{array}$.

Cut = up-rule below down-rule
= down-rule above up-rule

Corollary (Cut Elimination I)

For each proof $\begin{array}{c} \Pi\,{\downarrow,\uparrow} \\ B \end{array}$ there is a proof $\begin{array}{c} \Pi\,{\downarrow} \\ B \end{array}$.

Corollary (Craig Interpolation)

If $A \Rightarrow B$ then $\exists C.$ $A \Rightarrow C \;\wedge$
$C \Rightarrow B \;\wedge$
$\ell(C) \subseteq \ell(A) \cap \ell(B)$.

Cut Elimination I via Sequent Calculus

Theorem For each proof $\underset{A}{\Vert}{}^{\uparrow,\downarrow}$ there is one $\underset{A}{\Vert}{}^{\downarrow}$.

Proof

$$\underset{A}{\Vert}{}^{\downarrow,\uparrow} \rightsquigarrow \underset{\vdash A}{\nabla}{}^{GS1+\mathrm{cut}} \xrightarrow{\text{Gentzen}} \underset{\vdash A}{\nabla}{}^{GS1} \rightsquigarrow \underset{A}{\Vert}{}^{\downarrow}$$

example:

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \wedge B} \rightsquigarrow \mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{([\Gamma, A], [\Delta, B])}{[\Gamma, (A, [\Delta, B])]}}{[\Gamma, \Delta, (A, B)]}$$

$$\begin{array}{c} \rho\,\dfrac{S\{R\}}{S\{T\}} \\ \Vert \\ U \end{array} \rightsquigarrow \begin{array}{c} \nabla \\ \vdash T, \bar{R} \\ \nabla \\ \mathrm{cut}\,\dfrac{\vdash S\{T\}, \overline{S\{R\}} \qquad \vdash S\{R\}}{\vdash S\{T\}} \\ \nabla \\ \vdash U \end{array}$$

$\square$

Proof of Cut Elimination II

use two-sided sequent calculus like LK but:

- formulas restricted to negation normal form
- no negation rules
- axioms: $a \vdash a$
  $a, \bar{a} \vdash$
  $\vdash a, \bar{a}$

$$
\begin{array}{c} A \\ \| {\scriptstyle \uparrow,\downarrow} \\ B \end{array}
\rightsquigarrow
\underset{A \vdash B}{\nabla}\ \text{w/ cuts}
\xrightarrow{\text{cut elimination}}
\underset{A \vdash B}{\nabla}\ \text{cut-free}
\rightsquigarrow
\begin{array}{c} A \\ \| \uparrow \\ C \\ \| \downarrow \\ B \end{array}
$$

$$a \vdash a \rightsquigarrow a$$

$$a, \bar{a} \vdash \;\rightsquigarrow\; i\!\uparrow \frac{(a,\bar{a})}{f} \qquad \vdash a, \bar{a} \;\rightsquigarrow\; i\!\downarrow \frac{t}{[a,\bar{a}]}$$

$$
c_L \frac{\begin{array}{c}\nabla \\ \Phi, A, A \vdash \Psi\end{array}}{\Phi, A \vdash \Psi}
\rightsquigarrow
\begin{array}{c} c\!\uparrow \dfrac{\Phi, A}{(\Phi, A, A)} \\ \| \uparrow \\ C \\ \| \downarrow \\ \Psi \end{array}
\qquad
\frac{\begin{array}{c}\nabla \\ \Phi \vdash A, A, \Psi\end{array}}{\Phi \vdash A, \Psi}
\rightsquigarrow
\begin{array}{c} \Phi \\ \| \uparrow \\ C \\ \| \downarrow \\ \dfrac{[A, A, \Psi]}{[A, \Psi]} \end{array}
$$

$$
\wedge_R \; \frac{\Phi \vdash A, \Psi \qquad \Phi' \vdash B, \Psi'}{\Phi, \Phi' \vdash A \wedge B, \Psi, \Psi'}
\quad \rightsquigarrow \quad
\begin{array}{c}
(\Phi \;,\; \Phi') \\
\Vert\!\uparrow \quad \Vert\!\uparrow \\
(C \;,\; C') \\
\Vert\!\downarrow \quad \Vert\!\downarrow \\
s^2 \, \dfrac{([A, \Psi]\;[B, \Psi'])}{[(A, B), \Psi, \Psi']}
\end{array}
$$

$$
\mathrm{Cut} \; \frac{\Phi \vdash A, \Psi \qquad \Phi', A \vdash \Psi'}{\Phi, \Phi' \vdash \Psi, \Psi'}
\quad \rightsquigarrow \quad
\begin{array}{c}
(\Phi' \;,\; \Phi) \\
\qquad \Vert\!\uparrow \\
(\Phi' \;,\; C) \\
\qquad \Vert\!\downarrow \\
s \, \dfrac{(\Phi' \;,\; [A, \Psi])}{[\Psi, (\Phi', A)]} \\
\Vert\!\uparrow \\
[\Psi \;,\; C'] \\
\Vert \\
[\Psi \;,\; \Psi']
\end{array}
$$

PROBLEM 1 Find an internal cut elimination II procedure, i.e. without reference to sequent calculus.

PROBLEM 2 Prove cut elimination II for logics without cut-free sequent calculus, like BV, or modal logics like B, K5, S5.

PROBLEM 3 Computational interpretation, intuitionistic logic