Syntactic Cut Elimination for an Infinitary System for Common Knowledge

Kai Brünnler
(joint work with Thomas Studer)

Outline

The Logic of Common Knowledge

Formulas

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond_i A \mid \Box_i A \mid \circledast A \mid \boxast A$$

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Everybody knows

$$\Box A = \Box_1 A \wedge \ldots \wedge \Box_h A \quad \text{and} \quad \Diamond A = \Diamond_1 A \vee \ldots \vee \Diamond_h A.$$

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Notation

$$\Box^n A = \underbrace{\Box \ldots \Box}_{n-\text{times}} A$$

A Hilbert System

System $\mathsf{H_C}$

$$
\begin{array}{ll}
(\mathrm{K}) & \Box_i A \wedge \Box_i (A \supset B) \supset \Box_i B \\
(\mathrm{CCL}) & \boxast A \supset (\Box A \wedge \Box \boxast A)
\end{array}
$$

$$
(\mathrm{IND})\ \frac{B \supset (\Box A \wedge \Box B)}{B \supset \boxast A} \qquad (\mathrm{NEC})\ \frac{A}{\Box_i A}
$$

$$
(\mathrm{MP})\ \frac{A \quad A \supset B}{B}
$$

An Infinitary Sequent Calculus

(Alberucci and Jäger 2005)

System $\mathsf{G_C}$

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box_i\,\frac{\Gamma, \circledast\Delta, A}{\Diamond_i\Gamma, \circledast\Delta, \Box_i A, \Sigma}$$

$$\boxast\,\frac{\Gamma, \Box^k A \quad \text{for all } k \geq 1}{\Gamma, \boxast A} \qquad \circledast\,\frac{\Gamma, \circledast A, \Diamond A}{\Gamma, \circledast A}$$

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The Problem for Cut Elimination

$$\text{cut}\ \dfrac{\Box_i\ \dfrac{\begin{matrix}\pi_1\\ A, \Gamma, \diamondast \bar{B}\end{matrix}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \diamondast \bar{B}} \qquad \boxast\ \dfrac{\vdots \quad \begin{matrix}\pi_{2k}\\ \Box^k B, \Delta\end{matrix} \quad \vdots}{\boxast B, \Delta}\ 1 \leq k < \omega}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta}$$

The Problem for Cut Elimination

$$
\dfrac{
\dfrac{\dfrac{\pi_1}{A, \Gamma, \diamondast \bar{B}}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \diamondast \bar{B}}\;\Box_i
\qquad
\dfrac{\vdots \quad \dfrac{\pi_{2k}}{\Box^k B, \Delta} \quad \vdots \qquad 1 \leq k < \omega}{\boxast B, \Delta}\;\boxast
}{
\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta
}\;\text{cut}
$$

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$$
\text{cut}\;\dfrac{\Box_i\;\dfrac{\overset{\pi_1}{A, \Gamma, \circledast \bar{B}}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \circledast \bar{B}} \qquad \boxast\;\dfrac{\vdots \quad \overset{\pi_{2k}}{\Box^k B, \Delta} \quad \vdots}{\boxast B, \Delta}\; 1 \le k < \omega}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta}
$$

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$$
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$$

$$
\rightsquigarrow \quad
\dfrac{\dfrac{\overset{\pi_1}{A, \Gamma, \circledast \bar{B}} \qquad \dfrac{\cdots \quad \overset{\pi_{2k}}{\Box^k B, \Delta} \quad \cdots}{\boxast B, \Delta}\;\boxast \quad 1 \leq k < \omega}{A, \Gamma, \Delta}\;\text{cut}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Diamond_i \Delta}\;\Box_i
$$

Some Background on Deep Inference

Deep Inference in General

- Schütte 1950

Deep Sequents in Particular

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- Kashima 1994, Tanaka 2003
- Brünnler 2006
- Kretz and Studer 2006
- Poggiolesi (PhD student in Florence)

Deep Sequents

Definition

A *(deep) sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]_i$ where Γ is a sequent and $1 \leq i \leq h$.

Notation

As usual, given two sequents Γ and Δ, the comma in the expression Γ, Δ is multiset union. As usual, we write sequents *without curly braces*.

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Notation

As usual, given two sequents Γ and Δ, the comma in the expression Γ, Δ is multiset union. As usual, we write sequents *without curly braces*. So we write

$$A, [[B]_1, C, [D]_2]_3$$

instead of

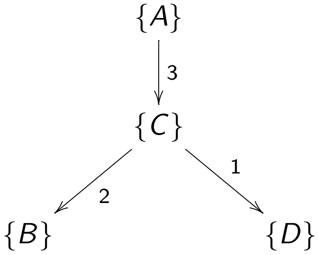
$$\{A, [\{[\{B\}]_1, C, [\{D\}]_2\}]_3\} \qquad .$$

Deep Sequents

Deep sequents have a *tree structure*, for example

$$A, [[B]_1, C, [D]_2]_3$$

has:



Fact

A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1]_{i_1}, \ldots, [\Delta_n]_{i_n} \quad ,$$

where the i_j range from 1 to h.

Definition

The *corresponding formula* of the above sequent is $\bot$ if $m = n = 0$ and otherwise

$$A_1 \vee \cdots \vee A_m \vee \Box_{i_1} D_1 \vee \cdots \vee \Box_{i_n} D_n \quad ,$$

where $D_1 \ldots D_n$ are the corresponding formulas of the sequents $\Delta_1 \ldots \Delta_n$.

Sequent Contexts

Definition

A *context* is a sequent with exactly one occurrence of the symbol $\{\ \}$, the *hole*, which does not occur inside formulas. Such contexts are denoted by $\Gamma\{\ \}$, $\Delta\{\ \}$, and so on. The hole is also called the *empty context*. The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\ \}$ inside $\Gamma\{\ \}$ by Δ. For example, if

$$\Gamma\{\ \} = A, [[B]_1, \{\ \}]_3 \qquad \text{and} \qquad \Delta = C, [D]_2$$

then

$$\Gamma\{\Delta\} = A, [[B]_1, C, [D]_2]_3 \quad .$$

Sequent Contexts

1) The symbol $\{ \}$ is a *context*, it is the *empty context*.
2) If Δ is a sequent and $\Gamma\{ \}$ is a context then $\Delta, \Gamma\{ \}$ is a *context*.
3) If $\Gamma\{ \}$ is a context then $[\Gamma\{ \}]$ is a *context*.

For a context $\Gamma\{ \}$ and a sequent Δ, $\Gamma\{\Delta\}$ is defined as follows:
1) If $\Gamma\{ \} = \{ \}$ then $\Gamma\{\Delta\} = \Delta$.
2) If $\Gamma\{ \} = \Sigma, \Lambda\{ \}$ then $\Gamma\{\Delta\} = \Sigma, \Lambda\{\Delta\}$.
3) If $\Gamma\{ \} = [\Lambda\{ \}]$ then $\Gamma\{\Delta\} = [\Lambda\{\Delta\}]$.

A Deep Sequent System

$$\Gamma\{p, \bar{p}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\,\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box_i\,\frac{\Gamma\{[A]_i\}}{\Gamma\{\Box_i A\}} \qquad \Diamond_i\,\frac{\Gamma\{\Diamond_i A, [\Delta, A]_i\}}{\Gamma\{\Diamond_i A, [\Delta]_i\}}$$

$$\boxast\,\frac{\Gamma\{\Box^k A\} \quad \text{for all } k \geq 1}{\Gamma\{\boxast A\}} \qquad \circledast\,\frac{\Gamma\{\circledast A, \Diamond^k A\}}{\Gamma\{\circledast A\}}$$

Admissible Rules

$$\text{nec}\,\frac{\Gamma}{[\Gamma]_i}\qquad \text{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}}\qquad \text{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\text{cut}\,\frac{\Gamma\{A\}\qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Derivations and proofs.

- *derivation in a system $\mathcal{S}$*: a well-founded tree whose nodes are labelled with sequents and which is built according to the inference rules from $\mathcal{S}$
- *conclusion of a derivation*: the sequent at the root
- *the premises of a derivation*: the sequents at the leaves
- *proof of a sequent Γ in a system*: a derivation in this system with conclusion Γ where all leaves are axioms.
- $\mathcal{S} \vdash \Gamma$: there is a proof of Γ in system $\mathcal{S}$.

Formula rank

$$
\begin{aligned}
&rk(p) = rk(\bar{p}) = 0 \\
&rk(A \wedge B) = rk(A \vee B) = max(rk(A), rk(B)) + 1 \\
&rk(\Box_i A) = rk(\Diamond_i A) = rk(A) + 1 \\
&rk(\boxast A) = rk(\diamondast A) = \omega + rk(A)
\end{aligned}
$$

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&rk(\boxast A) = rk(\diamond\!\!\!\ast A) = \omega + rk(A)
\end{aligned}
$$

Lemma (Some properties of the rank)

(i) $rk(A) = rk(\bar{A})$,

(ii) there are $m, n < \omega$ *such that* $rk(A) = \omega \cdot m + n$,

(iii) for all $k < \omega$ *we have* $rk(\Box^k A) < rk(\boxast A)$.

Cut rank

- *cut rank of an instance of* cut: rank of its cut formula
- cut_γ: cut with at most rank γ
- $\mathsf{cut}_{<\gamma}$: cut with a rank strictly smaller than γ.
- $\mathcal{S} \vdash^{\alpha}_{\gamma} \Gamma$: there is a proof of Γ in system $\mathcal{S} + \mathsf{cut}_{<\gamma}$ with depth bounded by α.

Admissibility and invertibility.

- An inference rule ρ with premises $\Gamma_1, \Gamma_2 \ldots$ and conclusion Δ is *depth- and cut-rank-preserving admissible* for a system $\mathcal{S}$ if whenever $\mathcal{S} \vdash^{\alpha}_{\gamma} \Gamma_i$ for each premise Γ_i then $\mathcal{S} \vdash^{\alpha}_{\gamma} \Delta$.
- For each rule ρ there is its *inverse*, denoted by $\bar{\rho}$, which has the conclusion of ρ as its only premise and any premise of ρ as its conclusion.
- An inference rule ρ is *depth- and cut-rank-preserving invertible* for a system $\mathcal{S}$ if $\bar{\gamma}$ is depth- and cut-rank preserving admissible for $\mathcal{S}$.

Lemma (Admissibility of the structural rules)

For system $\mathsf{D_C}$ *the following hold:*
(i) The necessitation, weakening and contraction rules are depth- and cut-rank-preserving admissible.
(ii) All rules are depth- and cut-rank-preserving invertible.

Lemma (Admissibility of the structural rules)

For system $\mathsf{D_C}$ *the following hold:*
(i) The necessitation, weakening and contraction rules are depth- and cut-rank-preserving admissible.
(ii) All rules are depth- and cut-rank-preserving invertible.

Lemma (Admissibility of the general identity axiom)

For all contexts $\Gamma\{\ \}$ *and all formulas* A *we have*
$\mathsf{D_C} \vdash^{2\cdot rk(A)}_{0} \Gamma\{A, \bar{A}\}$.

Theorem (Embedding the Hilbert system)

If $\mathsf{H_C} \vdash A$ *then there are* $m, n < \omega$ *such that* $\mathsf{D_C} \mathrel{\substack{\omega\cdot m \\ \overline{} \\ \omega\cdot n}} A$.

Proof

$$(\mathrm{IND})\ \frac{B \supset (\Box A \wedge \Box B)}{B \supset \circledast A}$$

by induction hypothesis $\mathsf{D_C} \mathrel{\vert\!\frac{\omega\cdot m_1}{\omega\cdot n_1}} B \supset (\Box A \wedge \Box B)$
by invertibility $\mathsf{D_C} \mathrel{\vert\!\frac{\omega\cdot m_1}{\omega\cdot n_1}} \bar{B}, \Box B$ and $\mathsf{D_C} \mathrel{\vert\!\frac{\omega\cdot m_1}{\omega\cdot n_1}} \bar{B}, \Box A$.
by induction on k: for all $k \geq 1$ there is an $m_{2k} < \omega$ such that
$\mathsf{D_C} \mathrel{\vert\!\frac{\omega\cdot m_1 + m_{2k}}{\omega\cdot n}} \bar{B}, \Box^k A$

Proof (continued)

$$
\text{cut}\ \dfrac{\bar{B}, \Box B \qquad \bigwedge \dfrac{\vdots \qquad \vee, \text{wk}\ \dfrac{\Box_i\ \dfrac{\Diamond_i, \text{wk}\ \dfrac{\text{nec}\ \dfrac{\bar{B}, \Box^k A}{[\bar{B}, \Box^k A]_i}}{\Diamond_i \bar{B}, [\Box^k A]_i}}{\Diamond_i \bar{B}, \Box_i \Box^k A}}{\Diamond \bar{B}, \Box_i \Box^k A} \qquad \vdots}{\Diamond \bar{B}, \Box^{k+1} A}\ {\scriptstyle 1 \leq i \leq h}}{\bar{B}, \Box^{k+1} A}
$$

$$
= \dfrac{\vee\ \dfrac{\boxast\ \dfrac{\bar{B}, \Box A \quad \ldots \quad \bar{B}, \Box^k A \quad \ldots}{\bar{B}, \boxast A}\ {\scriptstyle 1 \leq k < \omega}}{\bar{B} \vee \boxast A}}{B \supset \boxast A}
$$

$\Box$

Veblen Function

The *binary Veblen function* φ is generated inductively as follows:

1. $\varphi_0\beta := \omega^\beta$,
2. if $\alpha > 0$, then $\varphi_\alpha\beta$ denotes the βth common fixpoint of the functions $\lambda\xi.\varphi_\gamma\xi$ for $\gamma < \alpha$.

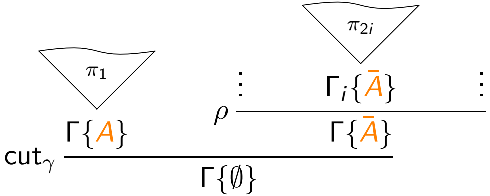
Lemma (Reduction Lemma)

If there is a proof

$$
\mathsf{cut}_\gamma\,\dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad\qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}
$$

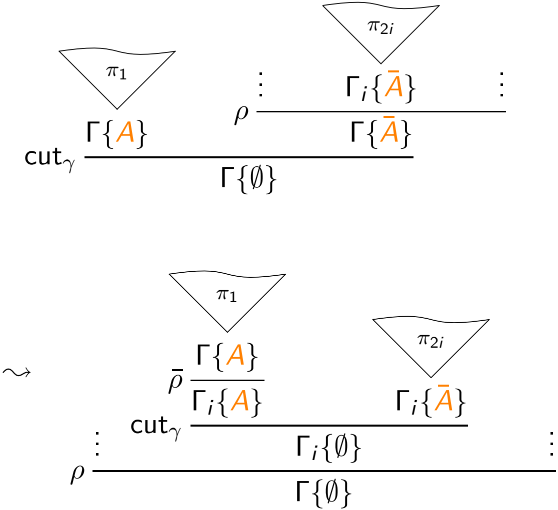
with π_1 *and* π_2 *in* $\mathsf{D_C} + \mathsf{cut}_{<\gamma}$ *then* $\vdash^{|\pi_1| \,\#\, |\pi_2|}_{\gamma} \Gamma\{\emptyset\}$ *.*

(passive):



π_1
π_{2i}
ρ
$\Gamma_i\{\bar{A}\}$
$\Gamma\{A\}$
$\Gamma\{\bar{A}\}$
cut_γ
$\Gamma\{\emptyset\}$

(passive):



$(\wedge - \vee)$:

$$\dfrac{\wedge\ \dfrac{\begin{matrix}\pi_{11} \\ \Gamma\{B\}\end{matrix} \qquad \begin{matrix}\pi_{12} \\ \Gamma\{C\}\end{matrix}}{\Gamma\{B \wedge C\}} \qquad \vee\ \dfrac{\begin{matrix}\pi_{21} \\ \Gamma\{\bar{B}, \bar{C}\}\end{matrix}}{\Gamma\{\bar{B} \vee \bar{C}\}}}{\Gamma\{\emptyset\}}\ \mathsf{cut}_{\sigma+1}$$

$(\wedge - \vee)$:

$$\mathsf{cut}_{\sigma+1}\ \dfrac{\wedge\ \dfrac{\begin{matrix}\pi_{11}\\ \Gamma\{B\}\end{matrix} \qquad \begin{matrix}\pi_{12}\\ \Gamma\{C\}\end{matrix}}{\Gamma\{B \wedge C\}} \qquad \vee\ \dfrac{\begin{matrix}\pi_{21}\\ \Gamma\{\bar{B}, \bar{C}\}\end{matrix}}{\Gamma\{\bar{B} \vee \bar{C}\}}}{\Gamma\{\emptyset\}}$$

$$\rightsquigarrow \quad \mathsf{cut}_{\sigma}\ \dfrac{\begin{matrix}\pi_{11}\\ \Gamma\{B\}\end{matrix} \qquad \mathsf{cut}_{\sigma}\ \dfrac{\mathsf{wk}\ \dfrac{\begin{matrix}\pi_{12}\\ \Gamma\{C\}\end{matrix}}{\Gamma\{\bar{B}, C\}} \qquad \begin{matrix}\pi_{21}\\ \Gamma\{\bar{B}, \bar{C}\}\end{matrix}}{\Gamma\{\bar{B}\}}}{\Gamma\{\emptyset\}}$$

$(\Box_i - \Diamond_i)$:

$$
\mathrm{cut}_{\sigma+1}\;
\dfrac{
\Box_i\,\dfrac{\begin{array}{c}\pi_{11}\\ \Gamma\{[\Delta]_i,[A]_i\}\end{array}}{\Gamma\{[\Delta]_i,\Box_i A\}}
\qquad
\Diamond\,\dfrac{\begin{array}{c}\pi_{21}\\ \Gamma\{[\Delta,\bar{A}]_i,\Diamond_i \bar{A}\}\end{array}}{\Gamma\{[\Delta]_i,\Diamond_i \bar{A}\}}
}{
\Gamma\{[\Delta]_i\}
}
$$

$(\Box_i - \Diamond_i)$:

$$\text{cut}_{\sigma+1}\ \dfrac{\Box_i\ \dfrac{\overset{\pi_{11}}{\vdots}\\ \Gamma\{[\Delta]_i,[A]_i\}}{\Gamma\{[\Delta]_i,\Box_i A\}} \qquad \Diamond\ \dfrac{\overset{\pi_{21}}{\vdots}\\ \Gamma\{[\Delta,\bar{A}]_i,\Diamond_i \bar{A}\}}{\Gamma\{[\Delta]_i,\Diamond_i \bar{A}\}}}{\Gamma\{[\Delta]_i\}}$$

$\leadsto$

$$\text{cut}_{\sigma}\ \dfrac{\text{ctr}\ \dfrac{\text{wk}^2\ \dfrac{\overset{\pi_{11}}{\vdots}\\ \Gamma\{[\Delta]_i,[A]_i\}}{\Gamma\{[\Delta,A]_i,[\Delta,A]_i\}}}{\Gamma\{[\Delta,A]_i\}} \qquad \text{cut}_{\sigma+1}\ \dfrac{\text{wk},\Box_i\ \dfrac{\overset{\pi_{11}}{\vdots}\\ \Gamma\{[\Delta]_i,[A]_i\}}{\Gamma\{[\Delta,\bar{A}]_i,\Box_i A\}} \qquad \overset{\pi_{21}}{\vdots}\\ \Gamma\{[\Delta,\bar{A}]_i,\Diamond_i \bar{A}\}}{\Gamma\{[\Delta,\bar{A}]_i\}}}{\Gamma\{[\Delta]_i\}}$$

$(\boxast - \diamondast)$:

$$
\dfrac{
\boxast\ \dfrac{\cdots \quad \overset{\pi_{1k}}{\Gamma\{\Box^k A\}} \quad \cdots \quad k<\omega}{\Gamma\{\boxast A\}}
\qquad
\diamondast\ \dfrac{\overset{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\diamondast \bar{A}\}}
}{\Gamma\{\emptyset\}}\ \mathrm{cut}_{\omega+\sigma}
$$

$(\boxast - \diamondast)$:

$$
\mathrm{cut}_{\omega+\sigma}\,\dfrac{\boxast\,\dfrac{\vdots \quad \dfrac{\pi_{1k}}{\Gamma\{\Box^k A\}} \quad \vdots}{\Gamma\{\boxast A\}}\;{\scriptstyle k<\omega} \qquad \diamondast\,\dfrac{\dfrac{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\diamondast \bar{A}\}}}{\Gamma\{\emptyset\}}
$$

$\rightsquigarrow$

$$
\mathrm{cut}_{\sigma+(j\cdot h)}\,\dfrac{\dfrac{\pi_{1j}}{\Gamma\{\Box^j A\}} \qquad \mathrm{cut}_{\omega+\sigma}\,\dfrac{\boxast\,\dfrac{\vdots \quad \mathrm{wk}\,\dfrac{\dfrac{\pi_{1k}}{\Gamma\{\Box^k A\}}}{\Gamma\{\Box^k A, \Diamond^j \bar{A}\}} \quad \vdots}{\Gamma\{\boxast A, \Diamond^j \bar{A}\}}\;{\scriptstyle k<\omega} \qquad \dfrac{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\Diamond^j \bar{A}\}}}{\Gamma\{\emptyset\}}
$$

First Elimination Lemma

If $\vdash_{\gamma+1}^{\alpha} \Gamma$ then $\vdash_{\gamma}^{2^{\alpha}} \Gamma$.

First Elimination Lemma

If $\vdash_{\gamma+1}^{\alpha} \Gamma$ then $\vdash_{\gamma}^{2^{\alpha}} \Gamma$.

Second Elimination Lemma

If $\vdash_{\beta+\omega^{\gamma}}^{\alpha} \Gamma$ then $\vdash_{\beta}^{\varphi_{\gamma}\alpha} \Gamma$.

First Elimination Lemma

If $\vdash^{\alpha}_{\gamma+1} \Gamma$ then $\vdash^{2^{\alpha}}_{\gamma} \Gamma$.

Second Elimination Lemma

If $\vdash^{\alpha}_{\beta+\omega^{\gamma}} \Gamma$ then $\vdash^{\varphi_{\gamma}\alpha}_{\beta} \Gamma$.

Theorem (Cut Elimination)

If A is a valid formula then $\vdash^{\varphi_2 0}_{0} A$.

Future work

- Extend the result to Alberucci/Jäger's system
- Extend the result to S5-based common knowledge
- Extend the result to the μ-calculus
- Find lower bounds
- Do syntactic cut elimination in a finitary system