Towards Cut Elimination for Temporal Logic

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joint work with Martin Lange

$$\text{Cut}\ \frac{A \supset B \qquad \bar{A} \supset B}{B}$$

$$\text{Induction}\ \frac{A(0) \qquad A(n) \supset A(n+1)}{\forall n.\, A(n)}$$

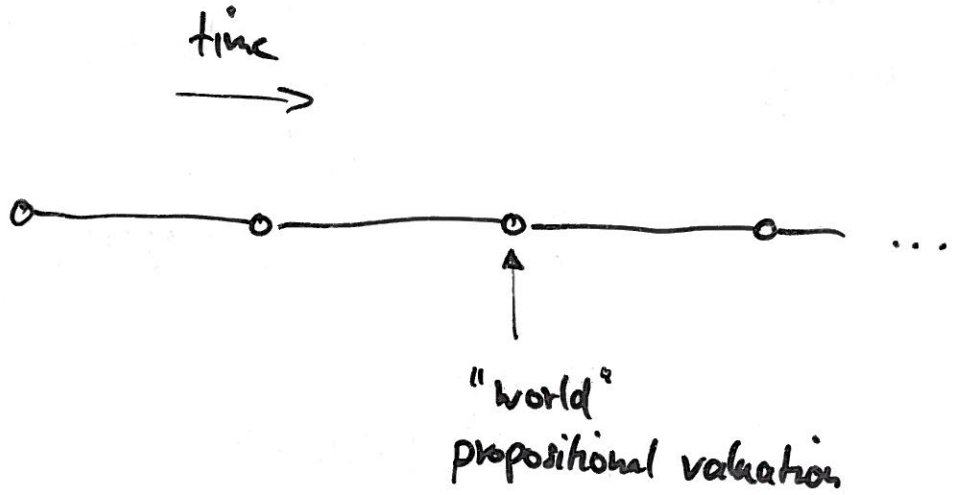
Temporal Logic

A concrete and simple one,
a fragment of LTL :

propositional connectives : $\wedge, \vee, \neg, \supset$ ...

temporal connectives :

- $\bigcirc$ - next
- $\Box$ - always
- $\Diamond$ - eventually



Some Valid Formulas

$\Box A \supset A$

$\Box A \supset \circ \Box A$

$\Box (A \supset B) \supset (\Box A \supset \Box B)$

$\Box (A \supset \circ A) \supset (A \supset \Box A)$ - Induction Axiom

add $\dfrac{A}{\Box A}$, $\dfrac{A \quad A \supset B}{B}$, prop. axioms

$\Rightarrow$ Axiomatisation
(i.e. complete)

×H

Fixpoints

$\Diamond A$

least solution to

$X \equiv A \vee \bigcirc X$

denoted

$\mu x.\ A \vee \bigcirc x$

approximated as

$\bot$

$A \vee \bigcirc \bot$

$A \vee \bigcirc (A \vee \bigcirc \bot)$

$A \vee \bigcirc (A \vee \bigcirc (A \vee \bigcirc \bot))$

$\vdots$

$\Box A$

greatest solution to

$X \equiv A \wedge \bigcirc X$

denoted

$\nu x.\ A \wedge \bigcirc x$

approximated as

$\top$

$A \wedge \bigcirc \top$

$A \wedge \bigcirc (A \wedge \bigcirc \top)$

$A \wedge \bigcirc (A \wedge \bigcirc (A \wedge \bigcirc \top))$

$\vdots$

"Proof Theory" ≙

Cut-Free Sequent Calculus

$$\Gamma, a, \bar{a} \qquad \wedge\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\text{cut}\ \frac{\Gamma, A \qquad \Delta, \bar{A}}{\Gamma, \Delta}$$

For LTL? "the naive proof system"

$$\Box\ \frac{A, \Gamma \qquad \circ\Box A, \Gamma}{\Box A, \Gamma} \qquad \Diamond\ \frac{\Gamma, A, \circ\Diamond A}{\Gamma, \Diamond A}$$

$$\circ\ \frac{\Gamma}{\circ\Gamma, \Sigma}$$

Incomplete

Induction axiom not provable

SAT

Ü

"Sequent Calculi" for LTL (complete)

1. Infinitary [Kawai, Szałas]

$$\Box \frac{\Gamma, A \quad \Gamma, \circ A \quad \Gamma, \circ\circ A \quad \dots}{\Gamma, \Box A}$$

2. "built-in" cut [Paech, Gudzius Kas]

$$\Box \frac{\Gamma, B \qquad \overline{B}, \circ B \qquad \overline{B}, A}{\Gamma, \Box A}$$

3. "cut-off" infinitary rule

$$\Box^k \frac{\Gamma, A \quad \Gamma, \circ A \ \dots \ \Gamma, \circ^k A}{\Gamma, \Box A}$$

sound for endsequents Γ' if $k \geq 2^{4|\Gamma'|}$

WANTED :

4. Cut-free, finitary,

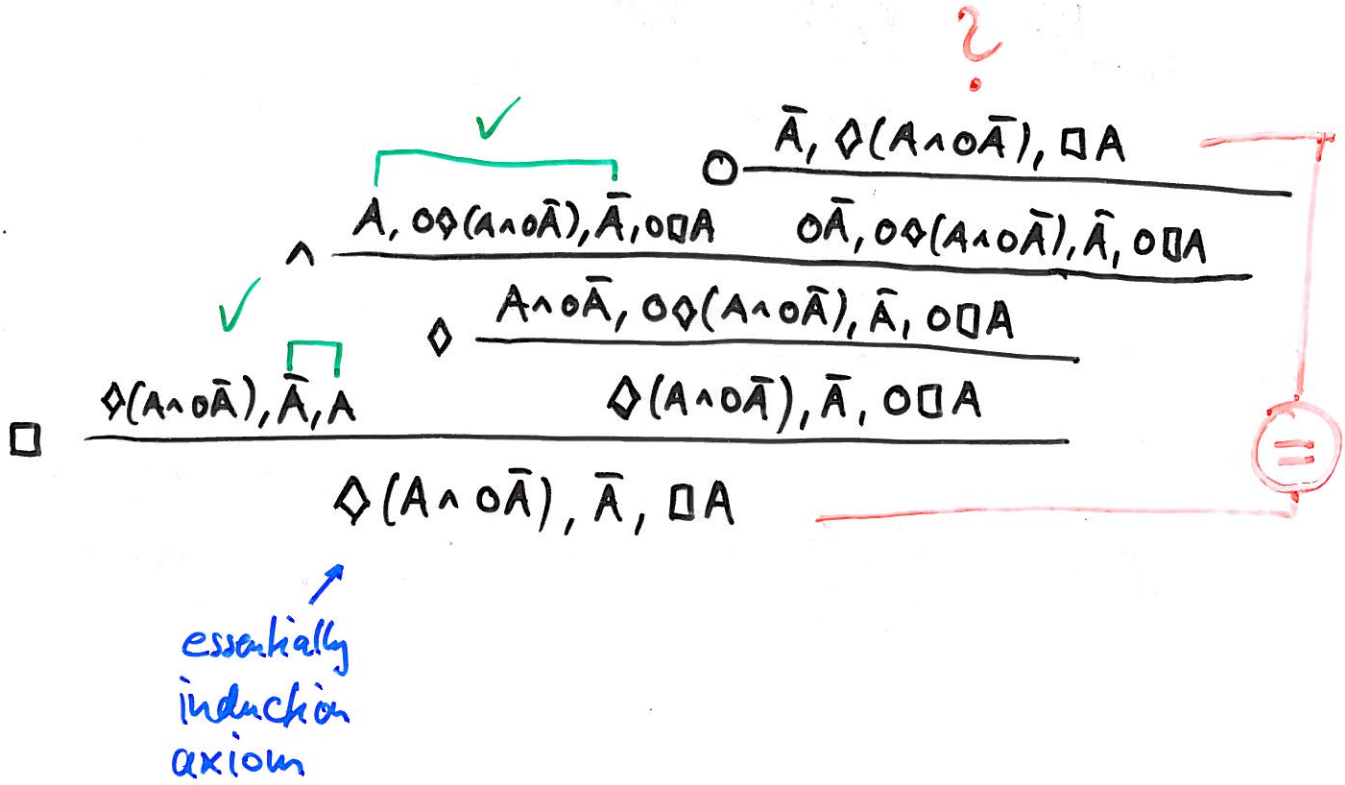
constant small number of premises,

"truly syntactic"

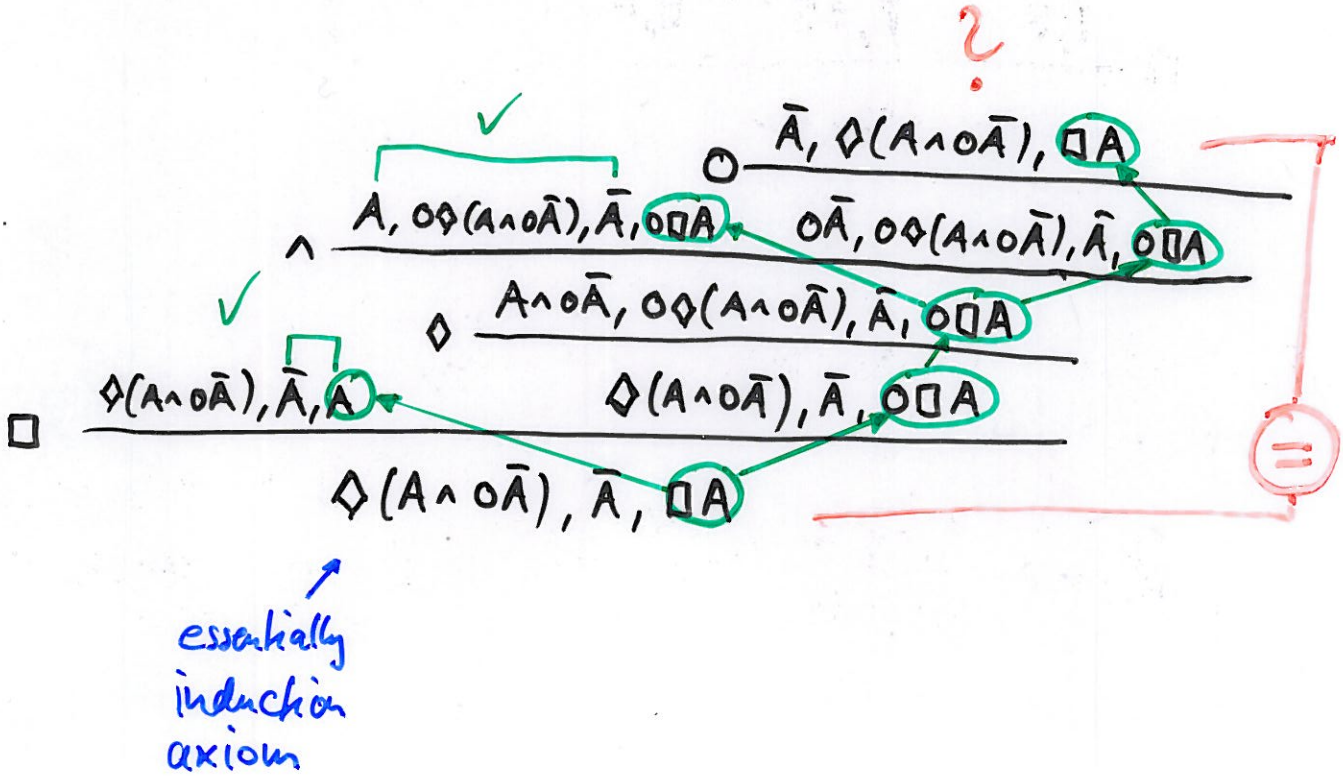
sequent system

So let's see what goes wrong in the naive system ...

A Failed Proof Attempt



A Failed Proof Attempt



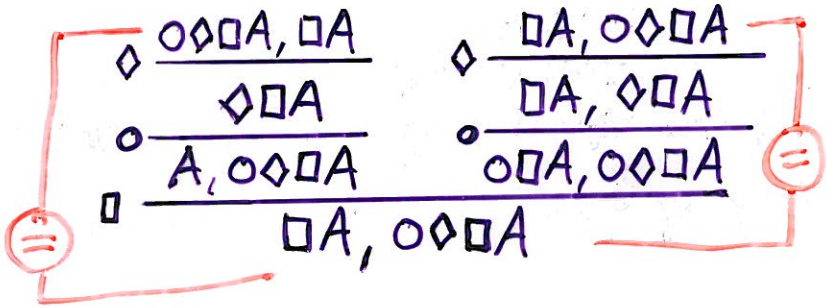
essentially induction axiom

non-solution #1 : add the induction axiom to the sequent calculus

→ still incomplete

non-solution #2 : consider a sequent axiomatic whenever this sequent already occurs below

→ unsound



Solution (Lange & Stirling, LICS 2007) :

Consider sequent axiomatic if it occurs below and a □-formula is connected to its other occurrence in the flow graph.

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→ still incomplete

non-solution #2 : consider a sequent axiomatic whenever this sequent already occurs below

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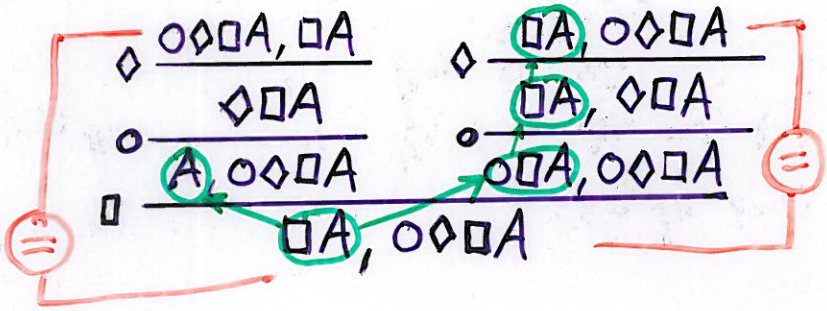


Solution (Lange & Stirling, LICS 2007) :

Consider sequent axiomatic if it occurs below and a □-formula is connected to its other occurrence in the flow graph.

A Cut-Free Sequent System

Annotated Formulas : $\Box_H A$

$$H = \{\Gamma_1, \ldots, \Gamma_n\}$$

set of sets of formulas

Sequents : sets of formulas with at most one annotated formula

naive sequent system +

$$\text{rep}\ \frac{}{\Box_{H,\Gamma} A, \Gamma} \qquad \Box_H\ \frac{\Gamma, A \qquad \Gamma, \bigcirc\Box_{H,\Gamma} A}{\Gamma, \Box_H A}$$

$$\text{foc}\ \frac{\Gamma, \Box_\emptyset A}{\Gamma, \Box A}$$

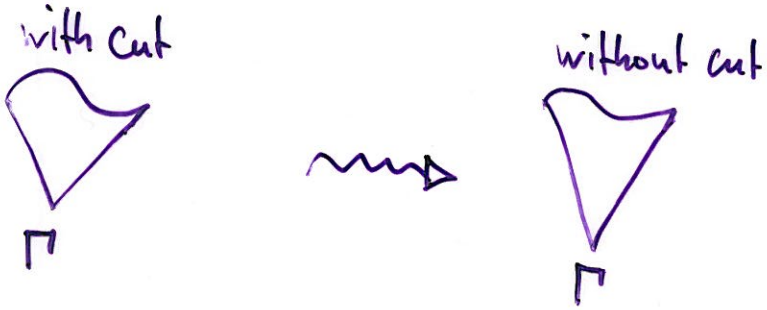
Completeness : very, very simple

Soundness : read $\Box_{\Gamma_1 \ldots \Gamma_n} A$ as $(\bar{\Gamma}_1 \vee \ldots \vee \bar{\Gamma}_n)\ \mathcal{R}\ (A \vee \bar{\Gamma}_1 \vee \ldots \vee \bar{\Gamma}_n)$

↑ "release"

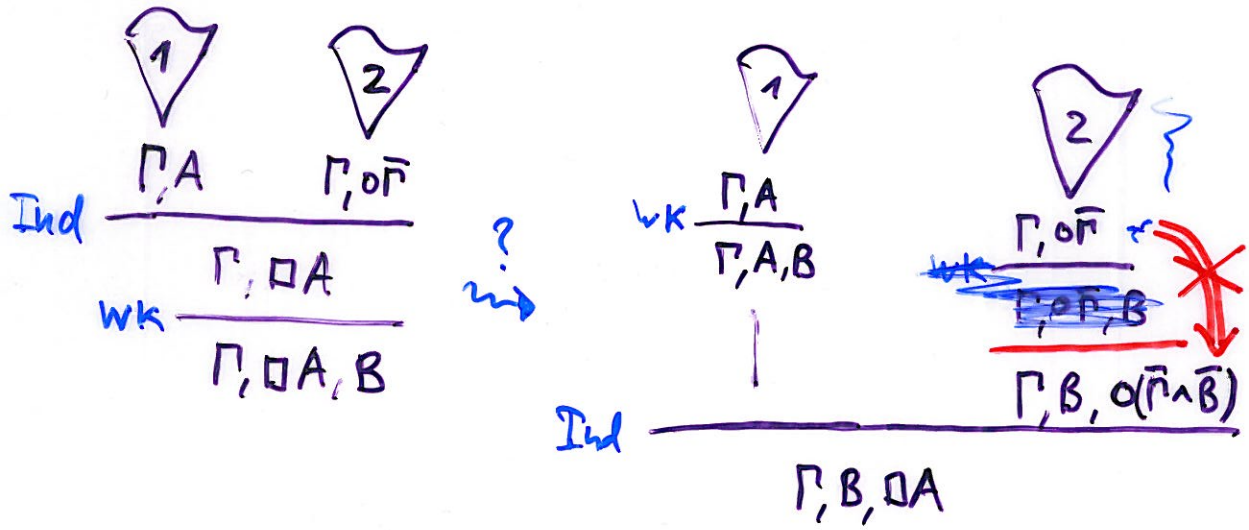
This also works for CTL (Lange & Stirling), PDL, Converse PDL, (Lange) Common Knowledge (Wehbe). Not yet CTL*, μ-calculus.

CURRENT PROBLEM : SYNTACTIC CUT ELIMINATION PROCEDURE

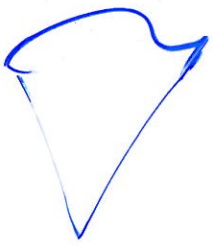


Cannot Eliminate Weakening

$$\text{Ind}\ \frac{\Gamma, A \qquad \Gamma, \circ\bar{\Gamma}}{\Gamma, \Box A} \qquad\qquad \text{Wk}\ \frac{\Gamma}{\Gamma, B}$$



$$\frac{A \wedge B, A \qquad A \wedge B, B}{A \wedge B}$$



$$\vdash \quad A_1 \sim A_n$$