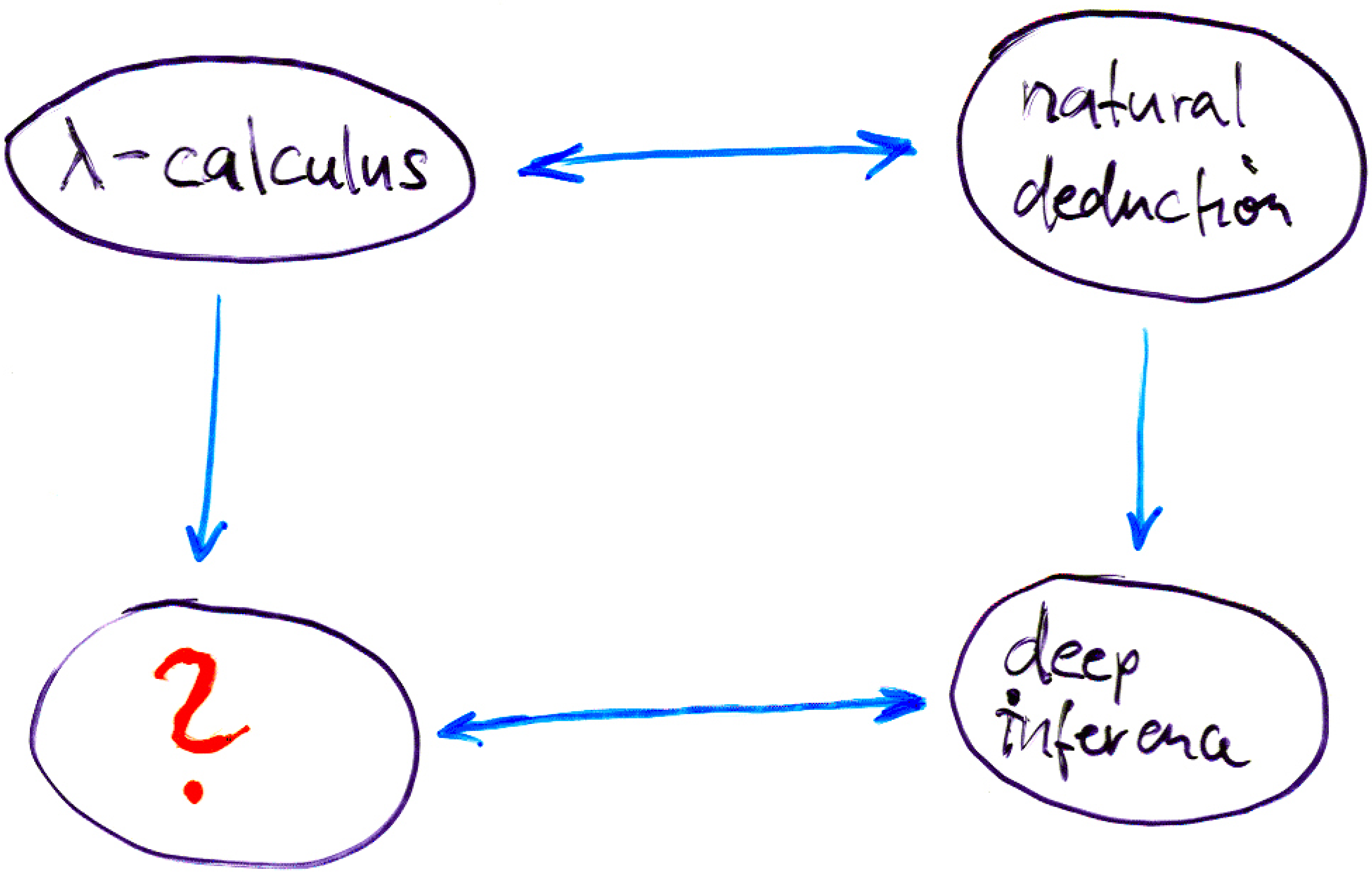
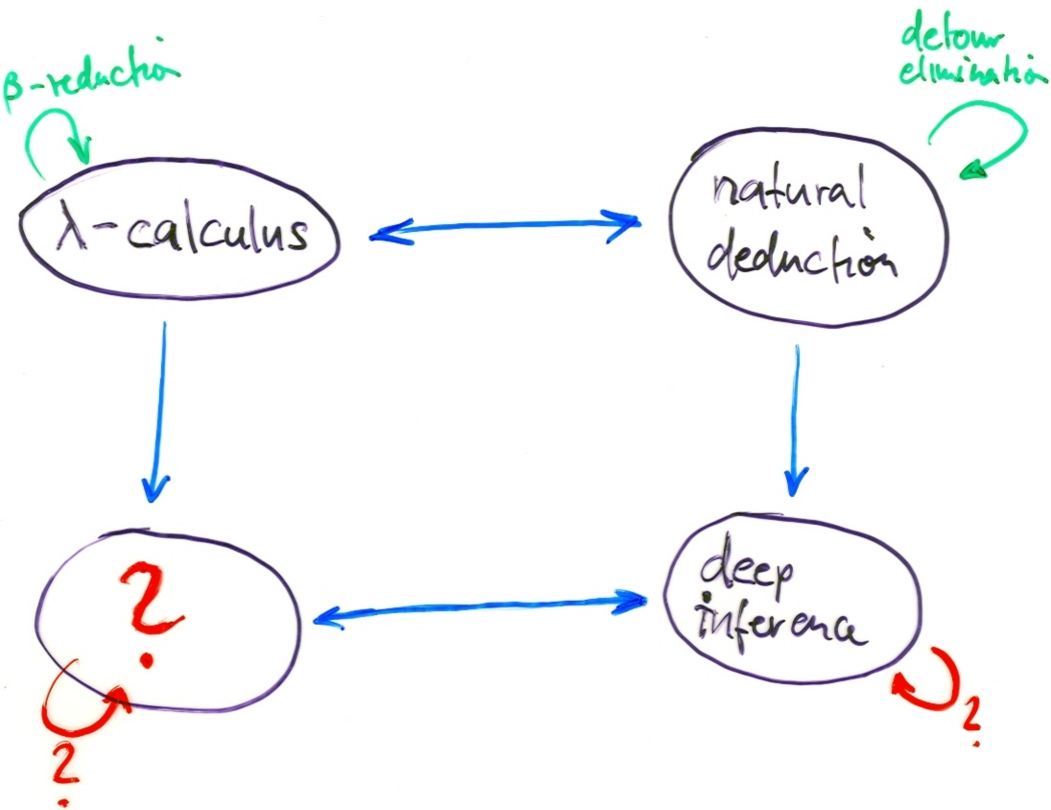

λ-calculus
natural deduction
?
deep inference


β-reduction
detour elimination
λ-calculus
natural deduction
?
deep inference
?
?

A Deep Inference System for Intuitionistic Logic

$$c\,\frac{A}{A\wedge A} \qquad\qquad i\,\frac{B}{A\supset(B\wedge A)}$$

$$w_1\,\frac{A\wedge B}{A} \qquad w_2\,\frac{A\wedge B}{B} \qquad e\,\frac{(A\supset B)\wedge A}{B}$$

Example Derivations

$$\dfrac{\dfrac{\dfrac{A\wedge B}{(A\wedge B)\wedge(A\wedge B)}\,c}{B\wedge(A\wedge B)}\,w_2}{B\wedge A}\,w_1 \qquad\qquad \dfrac{\dfrac{\dfrac{A\wedge B}{(B\supset(A\wedge B))\wedge B}\,i}{(B\supset A)\wedge B}\,w_1}{A}\,e$$

$$w_1\,\frac{A\supset C}{A\wedge B\supset C}$$

Proof Terms

$$R ::= id \mid p \in \{c, w_1, w_2, i, e\} \mid R.R \mid R \wedge R \mid R \supset R$$

Example Terms

$c.(w_2 \wedge id).(id \wedge w_1)$ $\quad$ $(i \wedge id).((id \supset w_1) \wedge id).e$

$w_1 \supset id$

Typing

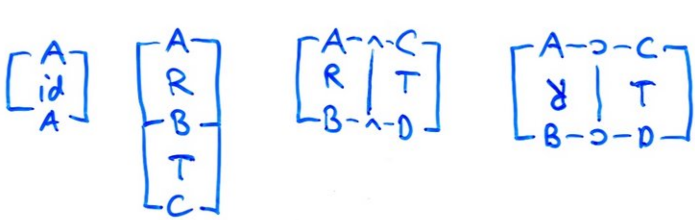
$A \xrightarrow{id} A$

$$\frac{A \xrightarrow{R} B \quad B \xrightarrow{T} C}{A \xrightarrow{R.T} C}$$

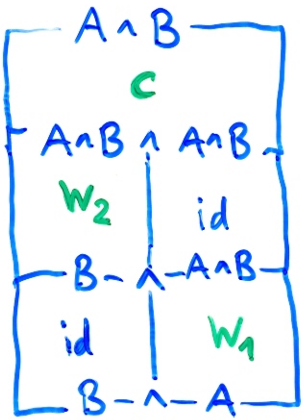
$$\frac{A \xrightarrow{R} B \quad C \xrightarrow{T} D}{A \wedge C \xrightarrow{R \wedge T} B \wedge D}$$

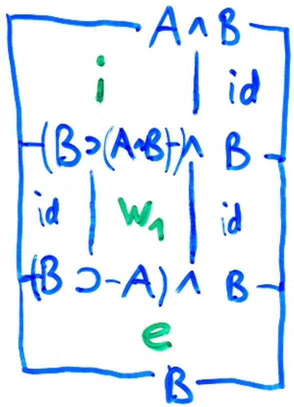
$$\frac{B \xrightarrow{R} A \quad C \xrightarrow{T} D}{A \supset C \xrightarrow{R \supset T} B \supset D}$$

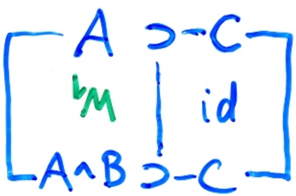
Typing Tiles



Typing Examples







Reduction

$$R.(T.U) \longrightarrow (R.T).U \qquad id \wedge id \to id$$

$$R.id \longrightarrow R \qquad id.R \longrightarrow R \qquad id \supset id \to id$$

$$(R \wedge T).(U \wedge V) \longrightarrow (R.U) \wedge (T.V)$$

$$(R \supset T).(U \supset V) \longrightarrow (U.R) \supset (T.V)$$

bur

nw
$$R \wedge T . w_1 \longrightarrow w_1 . R$$
$$R \wedge T . w_2 \longrightarrow w_2 . T$$

nc $\quad R.c \longrightarrow c.(R \wedge R)$

$\beta_\wedge \quad c.w_1 \to id \qquad c.w_2 \to id$

ni $\quad R.i \longrightarrow i.(id \supset (R \wedge id))$

subst

$\beta_\supset \quad (i.(id \supset R) \wedge T).e \longrightarrow (id \wedge T).R$

beta

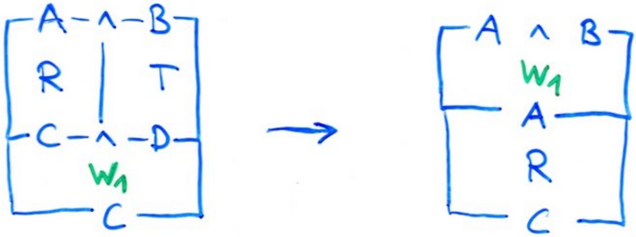
Reduction Examples

$$c.(w_2 \wedge id).(id \wedge w_1)$$
$$\to c.(w_2.id) \wedge (id.w_1)$$
$$\to^2 c.(w_2 \wedge w_1)$$

$$(i \wedge id).((id \supset w_1) \wedge id.e$$
$$\to (i.(id \supset w_1) \wedge (id.id)).e$$
$$\to (id \wedge id.id).w_1$$
$$\to^3 w_1$$

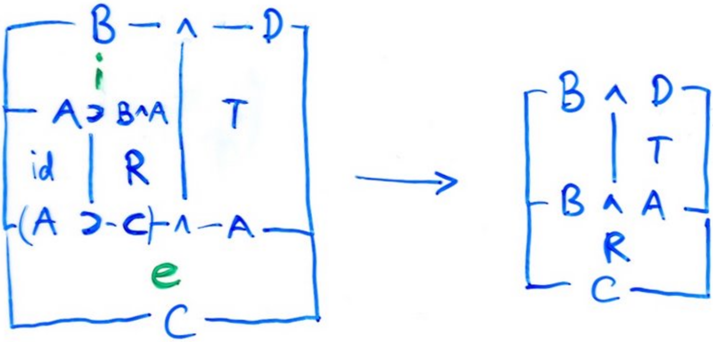
REDUCTION PRESERVES TYPING

nw



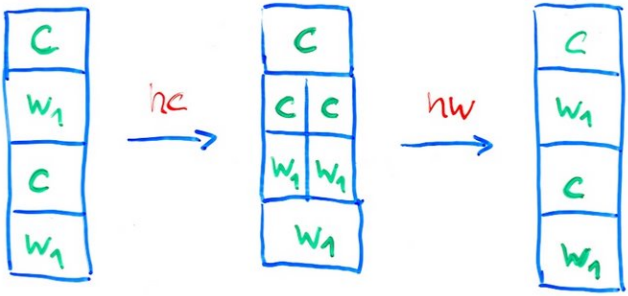
nc

$\beta_\supset$



We know : • the reduction rules are / can be made locally confluent

• not strongly normalising



We don't know : confluent ?

weakly normalising ?

strongly normalising under a certain strategy ?

(when typed !)

TRANSLATIONS

$$\begin{tikzcd} \text{Lambda Terms} \arrow[r, bend left, "D"] & \text{Deep Terms} \arrow[l, bend left, "L"] \end{tikzcd}$$

Theorems about D

preserves type : $\Gamma \vdash M : A \Rightarrow \begin{bmatrix} \bigwedge\Gamma \\ \underline{M}_D \\ A \end{bmatrix}$

preserves reduction : $M \twoheadrightarrow_\beta N \Rightarrow \underline{M}_D \twoheadrightarrow \underline{N}_D$

preserves normality : M normal $\Rightarrow$ $\underline{M}_D$ essentially normal

Theorems about L

preserves type : $\begin{bmatrix} A \\ R \\ B \end{bmatrix} \Rightarrow x : A \vdash \underline{R}_L : B$

.. no reduction / normality ...

Theorem (weak weak normalisation)

If $\begin{bmatrix} A \\ R \\ B \end{bmatrix}$ then $\exists T.\ \begin{bmatrix} A \\ T \\ B \end{bmatrix}$ and T normal.

left adj. functor	right adj. functor	Hom-bijection	unit	counit
$\Delta : f \mapsto (f,f)$	$\wedge : (f,g) \mapsto f \wedge g$	pairing $\langle -,- \rangle$	$c : A \to A \wedge A$	$w_1 : A \wedge B \to A$ $w_2 : A \wedge B \to B$
$- \wedge A : f \mapsto f \wedge id$	$A \supset - : f \mapsto id \supset f$	Currying $\Lambda(-)$	$i : B \to A \supset (B \wedge A)$	$e : (A \supset B) \wedge A \to B$

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Curien

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Curien

Richard & Kai

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Curien
Richard & Kai

$$\frac{A \xrightarrow{f} B \qquad A \xrightarrow{g} C}{A \xrightarrow{\langle f,g \rangle} B \wedge C}$$

$$\langle R, T \rangle := c . (R \wedge T)$$

$$\frac{B \wedge A \xrightarrow{f} C}{B \xrightarrow{\Lambda(f)} A \supset C}$$

$$\Lambda(R) := i . (id \supset R)$$