NESTED SEQUENTS

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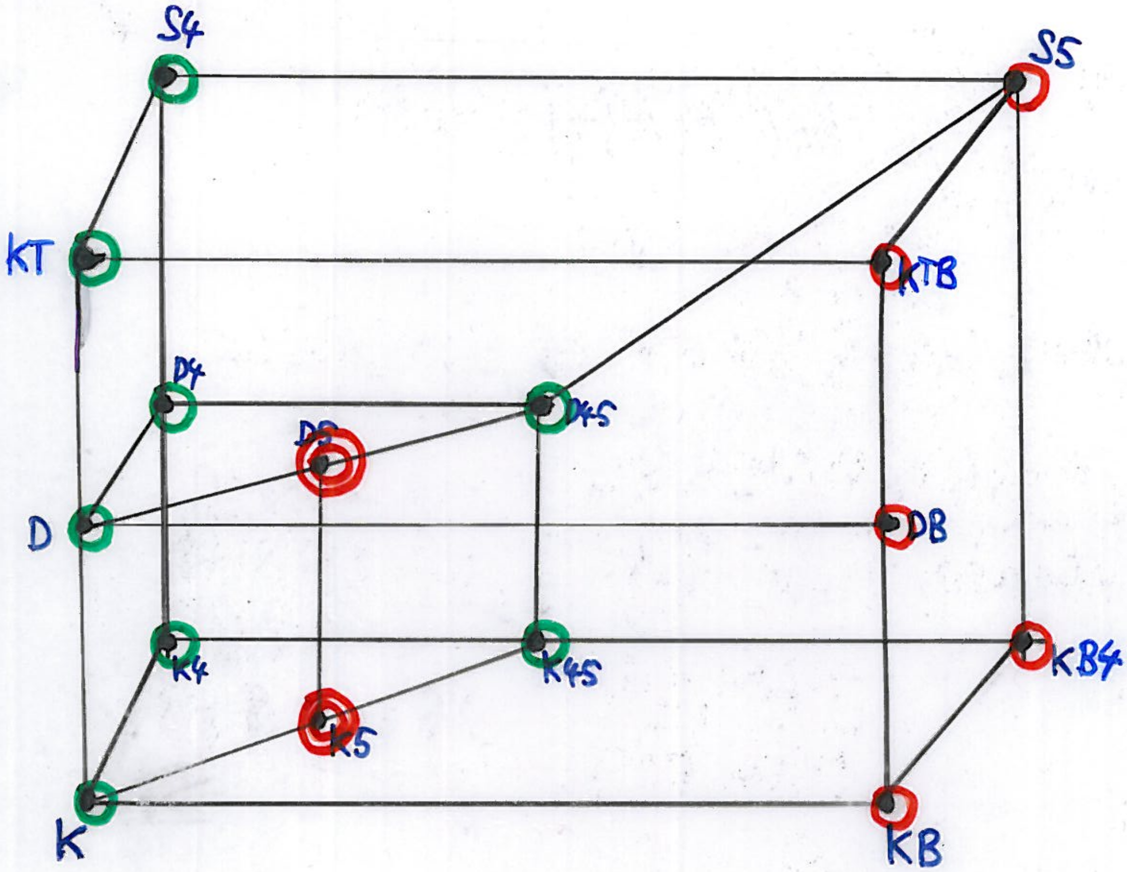
THE MODAL CUBE



there is a cut-free sequent calculus

no cut-free sequent calculus

not even analytic cut

Two Prominent Extensions of the Sequent Calculus

I. The Display Calculus

$$\mathrm{dp}\,\frac{X \circ Y \vdash Z}{Y \vdash (*X) \circ Z} \qquad \mathrm{dp}\,\frac{**X \vdash Y}{X \vdash Y} \qquad \mathrm{dp}\,\frac{\bullet X \vdash Y}{X \vdash \bullet Y}$$

$$\frac{\overleftarrow{\Diamond} X \supset Y}{X \supset \overrightarrow{\Box} Y}$$

II. Labelled Sequents

$$\Box\,\frac{\Gamma, tRs \vdash s{:}A, \Delta}{\Gamma \vdash t{:}\Box A, \Delta} \qquad 4\,\frac{\Gamma, sRt, tRu, sRu \vdash \Delta}{\Gamma, sRt, tRu \vdash \Delta}$$

(s is fresh)

d

Deep Sequents
(or Nested)

Def A *sequent* is a multiset of formulas and boxed sequents.

A *boxed sequent* is an expression $[\Gamma]$ where Γ is a sequent.

Example

$$A, [B], [C, [D, E]] \text{ is a sequent}$$

induces a tree:

$$\begin{tikzcd} & \{A\} \arrow[dl] \arrow[dr] & \\ \{B\} & & \{C\} \arrow[d] \\ & & \{D,E\} \end{tikzcd}$$

Def A *sequent context*, denoted $\Gamma\{\}$ is a sequent with exactly one occurrence of the symbol $\{\}$.

The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{\}$ by Δ in $\Gamma\{\}$.

S
b

Nested Sequents

History

- R.A. Bull 1992
- R. Kashima 1994
- myself 2006 "deep sequents"
- F. Poggiolesi 2007 "tree-hypersequents"

System K

$$\Gamma\{a,\bar{a}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

SYSTEM K

$$\Gamma\{a,\bar{a}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

FRAME CONDITIONS - $\Diamond$- RULES

$$\Diamond d\,\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}} \qquad \Diamond t\,\frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}} \qquad \Diamond b\,\frac{\Gamma\{A,[\Delta]\}}{\Gamma\{[\Diamond A,\Delta]\}}$$

$$\Diamond 4\,\frac{\Gamma\{[\Diamond A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}} \qquad \Diamond 5\,\frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\underbrace{\{\emptyset\}}_{>0}}$$

SYSTEM K

$$\Gamma\{a,\bar{a}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \mathsf{Ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

FRAME CONDITIONS AS STRUCTURAL RULES

$$\mathsf{d}\,\frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}\,\frac{\Gamma\{[\Sigma]\}}{\Gamma\{\Sigma\}} \qquad \mathsf{b}\,\frac{\Gamma\{[\Delta,[\Sigma]]\}}{\Gamma\{\Sigma,[\Delta]\}}$$

$$\mathsf{4}\,\frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,[\Sigma]]\}} \qquad \mathsf{5}\,\frac{\Gamma\{\emptyset\}\{[\Delta]\}}{\Gamma\{[\Delta]\}\{\emptyset\}}$$

70

System K

$$\Gamma\{a,\bar{a}\} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

Frame Conditions as Structural Rules

$$\mathsf{d}\,\frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}\,\frac{\Gamma\{[\Sigma]\}}{\Gamma\{\Sigma\}} \qquad \mathsf{b}\,\frac{\Gamma\{[\Delta,[\Sigma]]\}}{\Gamma\{\Sigma,[\Delta]\}}$$

$$\mathsf{4}\,\frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[\Delta,[\Sigma]]\}} \qquad \mathsf{5}\,\frac{\Gamma\{\emptyset\}\{[\Delta]\}}{\Gamma\{[\Delta]\}\{\emptyset\}}$$

>0

Theorem

For each $X \subseteq \{d,t,b,4,5\}$ K+X admits cut-elimination, and thus is complete.

Syntactic Cut Elimination

$$\mathsf{cut}\ \frac{\Gamma\{A\} \qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

$$\mathsf{cut}\ \dfrac{\Box\ \dfrac{\overset{1}{\vdots}}{\dfrac{\Gamma\{[\Delta],[A]\}}{\Gamma\{[\Delta],\Box A\}}} \qquad \Diamond\ \dfrac{\overset{2}{\vdots}}{\dfrac{\Gamma\{[\Delta,\bar{A}],\Diamond\bar{A}\}}{\Gamma\{[\Delta],\Diamond\bar{A}\}}}}{\Gamma\{[\Delta]\}} \quad \rightsquigarrow \quad \mathsf{cut}\ \dfrac{\mathsf{ctr}\ \dfrac{\mathsf{wk}^2\ \dfrac{\overset{1}{\vdots}}{\dfrac{\Gamma\{[\Delta],[A]\}}{\Gamma\{[\Delta,A],[\Delta,A]\}}}}{\Gamma\{[\Delta,A]\}} \qquad \mathsf{cut}\ \dfrac{\mathsf{wk}\ \dfrac{\overset{1}{\vdots}}{\dfrac{\Gamma\{[\Delta],[A]\}}{\Gamma\{[\Delta,\bar{A}],\Box A\}}} \qquad \dfrac{\overset{2}{\vdots}}{\Gamma\{[\Delta,\bar{A}],\Diamond\bar{A}\}}}{\Gamma\{[\Delta,\bar{A}]\}}}{\Gamma\{[\Delta]\}}$$