The Problem of Cut Elimination in Modal Predicate Logic

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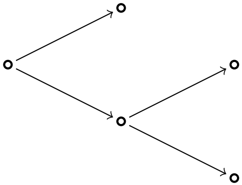
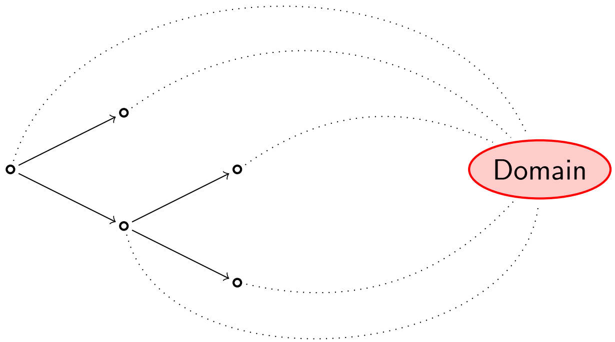
Modalities and Predicates

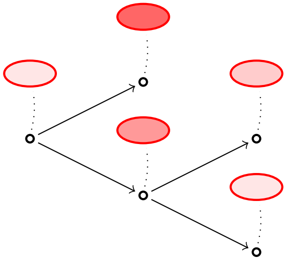


Modalities and Predicates



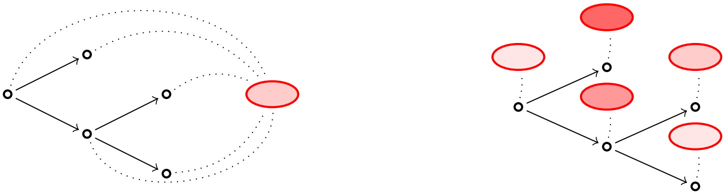
"Constant Domains"

Modalities and Predicates



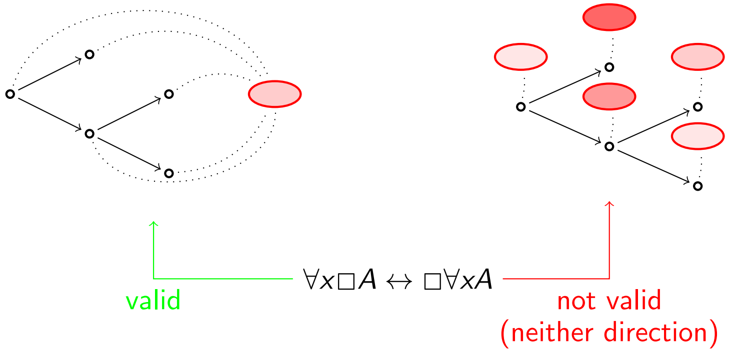
"Varying Domains"

The Barcan Formula



$$\forall x \Box A \leftrightarrow \Box \forall x A$$

The Barcan Formula



The Naive Proof System

	proof system for predicate logic
+	proof system for modal logic
=	constant domains? varying domains?

The Naive Proof System

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{ctr}\,\frac{\Gamma, A, A}{\Gamma, A}$$

$$\forall\,\frac{\Gamma, A[x := u]}{\Gamma, \forall x A}\ \begin{array}{c}u \text{ not free}\\ \text{in conclusion}\end{array} \qquad \exists\,\frac{\Gamma, A[x := u]}{\Gamma, \exists x A}$$

$$\Box\,\frac{\Gamma, A}{\Diamond\Gamma, \Box A, \Sigma}$$

Neither Varying nor Constant Domains...

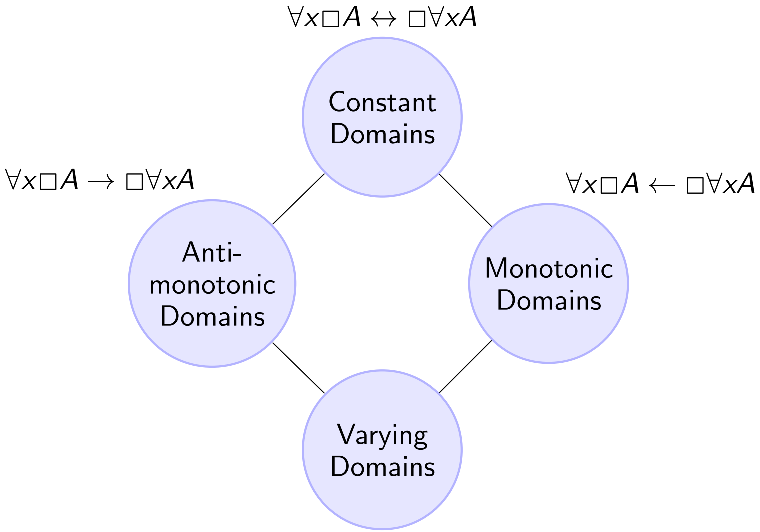
$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\bar{A}[x := u], A[x := z]}{\bar{A}[x := u], \forall x A}\,\forall}{\Diamond \bar{A}[x := u], \Box \forall x A}\,\Box}{\exists x \Diamond \bar{A}, \Box \forall x A}\,\exists}{\exists x \Diamond \bar{A} \vee \Box \forall x A}\,\vee}{\forall x \Box A \rightarrow \Box \forall x A}\,=
$$

Neither Varying nor Constant Domains...

$$=\dfrac{\vee\,\dfrac{\exists\,\dfrac{\Box\,\dfrac{\forall\,\dfrac{\bar{A}[x := u], A[x := z]}{\bar{A}[x := u], \forall x A}}{\Diamond\bar{A}[x := u], \Box\forall x A}}{\exists x\Diamond\bar{A}, \Box\forall x A}}{\exists x\Diamond\bar{A} \vee \Box\forall x A}}{\forall x\Box A \to \Box\forall x A}$$

$$=\dfrac{\vee\,\dfrac{\forall\,\dfrac{\Box\,\dfrac{\exists\,\dfrac{A[x := u], \bar{A}[x := u]}{A[x := u], \exists x\bar{A}}}{\Box A[x := u], \Diamond\exists x\bar{A}}}{\forall x\Box A, \Diamond\exists x\bar{A}}}{\forall x\Box A \vee \Diamond\exists x\bar{A}}}{\forall x\Box A \leftarrow \Box\forall x A}$$

More Domain Types



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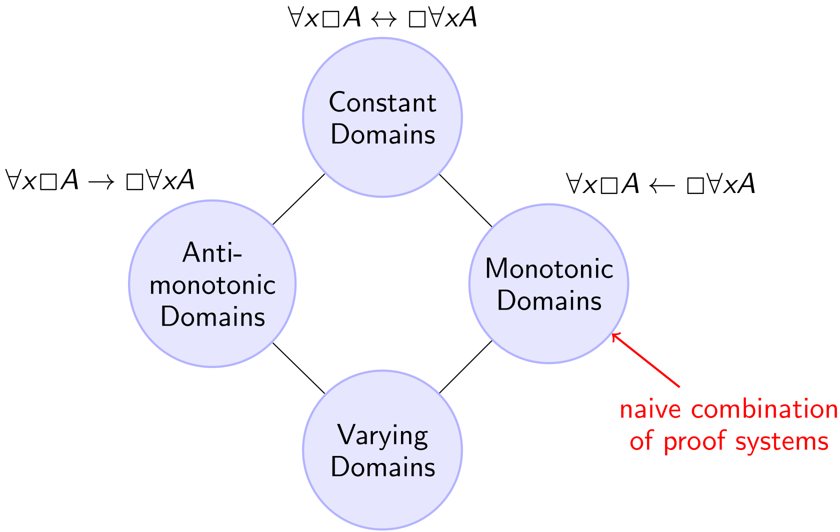

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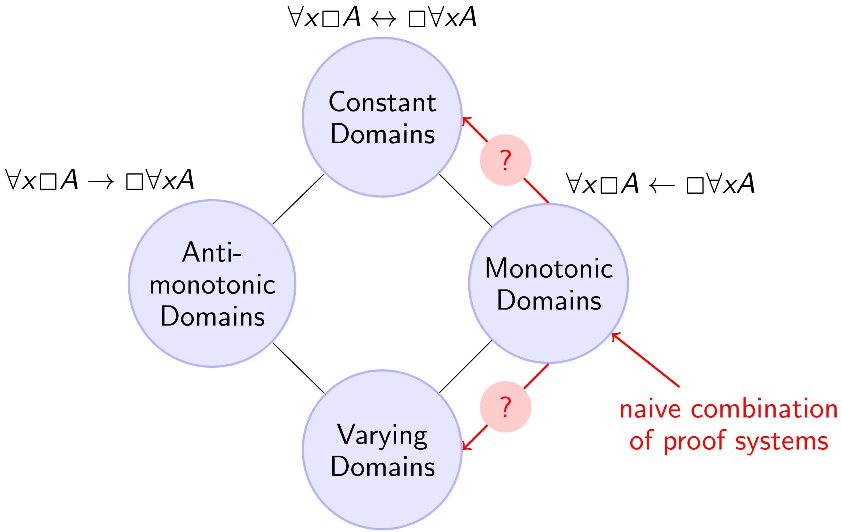

The Problem

Problem

Find a systematic cut-free axiomatisation of the four modal predicate logics.

Question

What goes wrong in the naive system?

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$$
\begin{array}{rc}
 & \bar{A}[x := u], A[x := z] \\ \cline{2-2}
\forall & \bar{A}[x := u], \forall x A \\ \cline{2-2}
\Box & \Diamond \bar{A}[x := u], \Box \forall x A \\ \cline{2-2}
\exists & \exists x \Diamond \bar{A}, \Box \forall x A \\ \cline{2-2}
\vee & \exists x \Diamond \bar{A} \vee \Box \forall x A \\ \cline{2-2}
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\end{array}
$$

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\begin{array}{rc}
 & A[x := u], \bar{A}[x := u] \\ \cline{2-2}
\exists & A[x := u], \exists x \bar{A} \\ \cline{2-2}
\Box & \Box A[x := u], \Diamond \exists x \bar{A} \\ \cline{2-2}
\forall & \forall x \Box A, \Diamond \exists x \bar{A} \\ \cline{2-2}
\vee & \forall x \Box A \vee \Diamond \exists x \bar{A} \\ \cline{2-2}
= & \forall x \Box A \leftarrow \Box \forall x A
\end{array}
$$

What goes wrong in the naive system?

$$= \; \dfrac{\vee \; \dfrac{\exists \; \dfrac{\Box \; \dfrac{\forall \; \dfrac{\bar{A}[x := u], A[x := {\color{red} z}]}{\bar{A}[x := u], \forall x A}}{\Diamond \bar{A}[x := u], \Box \forall x A}}{\exists x \Diamond \bar{A}, \Box \forall x A}}{\exists x \Diamond \bar{A} \vee \Box \forall x A}}{\forall x \Box A \to \Box \forall x A}$$

$$= \; \dfrac{\vee \; \dfrac{\forall \; \dfrac{\Box \; \dfrac{\exists \; \dfrac{A[x := u], \bar{A}[x := u]}{A[x := u], \exists x \bar{A}}}{\Box A[x := u], \Diamond \exists x \bar{A}}}{\forall x \Box A, \Diamond \exists x \bar{A}}}{\forall x \Box A \vee \Diamond \exists x \bar{A}}}{\forall x \Box A \leftarrow \Box \forall x A}$$

What goes wrong in the naive system?

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$$

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=\dfrac{\vee\dfrac{\forall\dfrac{\Box\dfrac{\exists\dfrac{A[x := u], \bar{A}[x := u]}{A[x := u], \exists x\bar{A}}}{\Box A[x := u], \Diamond\exists x\bar{A}}}{\forall x\Box A, \Diamond\exists x\bar{A}}}{\forall x\Box A \vee \Diamond\exists x\bar{A}}}{\forall x\Box A \leftarrow \Box\forall x A}
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What goes wrong in the naive system?

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$$=\;\dfrac{\vee\;\dfrac{\forall\;\dfrac{\Box\;\dfrac{\exists\;\dfrac{A[x:=u],\ \bar{A}[x:=u]}{A[x:=u],\ \exists x\bar{A}}}{\Box A[x:=u],\ \Diamond\exists x\bar{A}}}{\forall x\Box A,\ \Diamond\exists x\bar{A}}}{\forall x\Box A\vee\Diamond\exists x\bar{A}}}{\forall x\Box A\leftarrow\Box\forall x A}$$

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\qquad\qquad
=\dfrac{\vee\,\dfrac{\forall\,\dfrac{\Box\,\dfrac{\exists\,\dfrac{A[x:=u],\ \bar{A}[x:=u]}{A[x:=u],\ \exists x\bar{A}}}{\Box A[x:=u],\ \Diamond\exists x\bar{A}}}{\forall x\Box A,\ \Diamond\exists x\bar{A}}}{\forall x\Box A\vee\Diamond\exists x\bar{A}}}{\forall x\Box A\leftarrow\Box\forall x A}
$$

It’s not so clear how to solve our problem in the sequent system.
Let’s look at a Hilbert system, where it is easily solved.

A Hilbert System

All propositional tautologies, modus ponens, plus:

$$\Box(A \to B) \to (\Box A \to \Box B)$$

$$\text{Nec}\ \frac{A}{\Box A}$$

$$A \to \forall x A \qquad \text{if } x \text{ not free in } A$$

$$\forall x(A \to B) \to (\forall x A \to \forall x B)$$

$$\forall x A \to A[x := y]$$

$$\text{Gen}\ \frac{A}{\forall x A}$$

A Hilbert System

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$A \to \forall x A$ if x not free in A

$\forall x(A \to B) \to (\forall x A \to \forall x B)$

$\forall x A \to A[x := y]$ not valid for varying domains!

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$A \to \forall x A$ if x not free in A

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$\forall y(\forall x A \to A[x := y])$ valid for varying domains

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A Hilbert System

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$\Box(A \to B) \to (\Box A \to \Box B)$

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$A \leftrightarrow \forall x A$ if x not free in A

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$\forall y(\forall x A \to A[x := y])$ valid for varying domains

Gen $\dfrac{A}{\forall x A}$

$\forall x \forall y A \leftrightarrow \forall y \forall x A$

The Problem, again

Up to now was folklore, see for example the book "First-Order Modal Logic" by Fitting and Mendelsohn.

Problem

Find a systematic cut-free axiomatisation of the four modal predicate logics.

Subproblem

Find a cut-free system for predicate logic, which does not prove $\forall x A \rightarrow A[x := y]$.

Question

Can we get it in the same way we just got the Hilbert-system?

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Subproblem

Find a cut-free system for predicate logic, which does not prove $\forall x A \to A[x := y]$.

Question

Can we get it in the same way we just got the Hilbert-system?

How to weaken this system?

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{ctr}\,\frac{\Gamma, A, A}{\Gamma, A}$$

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Nested Sequents

Solution

Extend sequents by structural connective for $\forall x$, denoted $\forall x[\ ]$.

Idea

$\forall x[\ ]$ is for $\forall x$ what "," is for $\vee$.

Example

$$A, B \vee C, \forall x[A, B], \forall y[\forall z[\exists x E, \forall y F, G]]$$

Nested Sequents

Solution
Extend sequents by structural connective for $\forall x$, denoted $\forall x[\ ]$.

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A Nested Sequent System

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$$\forall\,\frac{\Gamma\{\forall x[A]\}}{\Gamma\{\forall x A\}} \qquad \mathsf{scp}\,\frac{\forall x[\Gamma\{\Delta\}]}{\Gamma\{\forall x[\Delta]\}}$$ where x does not occur in $\Gamma\{\ \}$

$$\exists_1\,\frac{\Gamma\{A[x := u]\}}{\Gamma\{\exists x A\}}$$ where u is not free in premise

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- sound for varying domains
  $\nvdash \forall x A \rightarrow A[x := y]$
- complete for sentences of predicate logic
  $\vdash \forall y(\forall x A \rightarrow A[x := y])$
- has syntactic cut-elimination
- easily captures empty domains
- has the "free variable property"

A Nested Sequent System

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